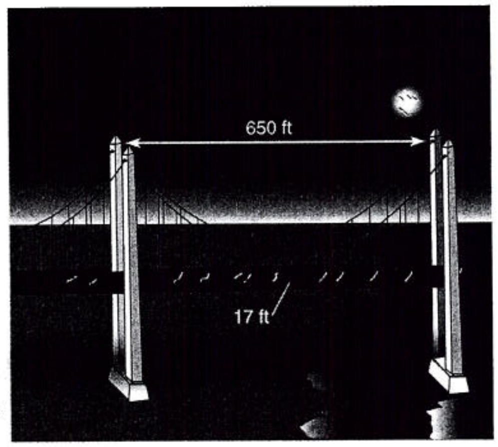
b. How far will the light source be from the rim of the searchlight?

48. A cable from a suspension bridge in the shape of a parabola hangs from two 75-feet towers, which are 650 feet apart. The lowest part of the cable hangs 17 feet from the roadway of the bridge.

a. Assume that the bridge cable will be supported by vertical beams. How long should such a support beam be if it is located 125 feet from the center of the span of the bridge?

b. Determine the length of vertical supporting beams for the bridge cable if they are to be placed (i) 75 feet, (ii) 200 feet, or (iii) 300 feet from the center of the cable.




# 5.4 The Ellipse and Hyperbola

## The Ellipse

The geometric definition of an ellipse is as follows:

### Definition of an Ellipse

An **ellipse** is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. Each fixed point is called a **focus** of the ellipse.

An ellipse may be constructed in the following way. Place a thumbtack at each of the foci $F_1$ and $F_2$, and attach one end of a string to each of the thumbtacks. Hold a pencil tightly against the string, as shown in Figure 24, and move the pencil. The point $P$ describes an ellipse since the sum of the distances from $P$ to the foci is always a constant, the length of the string.

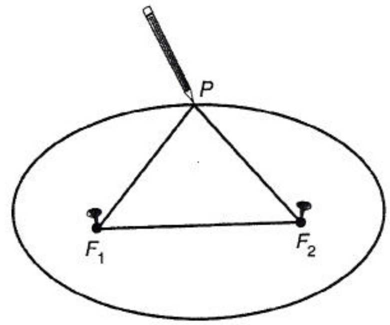



**FIGURE 24** The Foci of an Ellipse

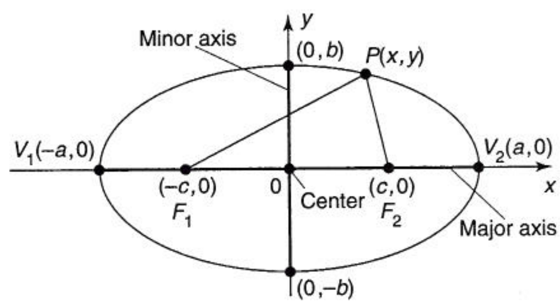



**FIGURE 25** Deriving the Equation of an Ellipse

The equation of the ellipse is in standard form if the two fixed points are on either the $x$-axis or the $y$-axis and they are equidistant from the origin. In Figure 25, we have placed these foci $F_1$ and $F_2$ on the $x$-axis. The line segment joining the foci is called the **major axis** of the ellipse, and the midpoint of the major axis is called the **center** of the ellipse. The line segment through the center and perpendicular to the major axis is called the **minor axis.** The points at which the ellipse intersects the major axis are called the **vertices** ($V_1$ and $V_2$ in Figure 25) of the ellipse. Note that the length of the major axis is $2a$, the length of the minor axis is $2b$, and the center of the ellipse is the origin. By placing the point $P$ at either $V_1$ or $V_2$, we see that "the length of the string" (See Figure 24.) is $2a$. (Why?) Therefore, we have that

$$\overline{F_1P} + \overline{F_2P} = 2a$$

which is to be true for all points $P$ on the ellipse. The equation of an ellipse in standard form is shown (See Exercise 35.) to be as follows.

**Standard Form of the Equation of an Ellipse with Horizontal Major Axis and Center (0, 0)**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b.$$

If we place foci $F_1$ and $F_2$ on the $y$-axis, symmetrically about the origin as shown in Figure 26, we can obtain the other standard form of the equation of an ellipse. Combining the above result, we have the following.

**Standard Form of the Equation of an Ellipse with Center (0, 0)**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \neq b$$

If $a > b$, the major axis is horizontal. If $a < b$, the major axis is vertical.

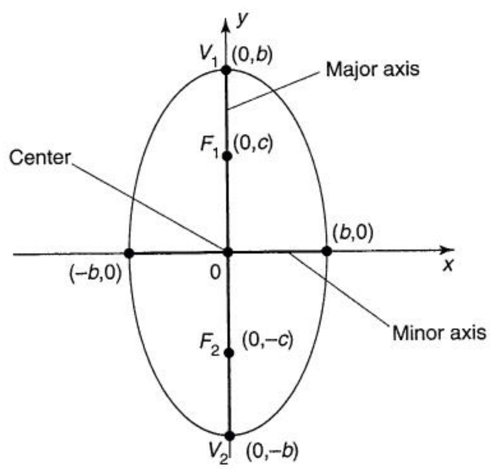



**FIGURE 26** Deriving the Equation of an Ellipse

Note that the equation indicates that the graph is symmetric with respect to the $x$-axis, the $y$-axis and the origin.

If we let $x = 0$ in the standard form, we find $y = \pm b$; if we let $y = 0$, we find $x = \pm a$. Thus, the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has intercepts $(\pm a, 0)$ and $(0, \pm b)$.

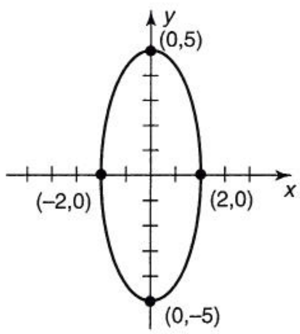



**FIGURE 27** Graph of $\frac{x^2}{4} + \frac{y^2}{25} = 1$

## EXAMPLE 1 USING THE EQUATION OF AN ELLIPSE

Find the intercepts and sketch the graph of the ellipse whose equation is

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

### SOLUTION

Setting $x = 0$ and solving for $y$ yields the $y$-intercepts $\pm 5$; setting $y = 0$ and solving for $x$ yields the $x$-intercepts $\pm 2$. The graph is shown in Figure 27.

## EXAMPLE 2 USING THE EQUATION OF AN ELLIPSE

Write the equation of the ellipse in standard form and determine the intercepts.

a. $4x^2 + 3y^2 = 12$ b. $9x^2 + y^2 = 10$

### SOLUTION

a. Dividing by 12 to make the right-hand side equal to 1, we have

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

The $x$-intercepts are $(\pm\sqrt{3}, 0)$; the $y$-intercepts are $(0, \pm 2)$.

b. Dividing by 10 we have

$$\frac{9x^2}{10} + \frac{y^2}{10} = 1$$

But this is *not* standard form. However, if we write

$$\frac{9x^2}{10} \quad \text{as} \quad \frac{\frac{x^2}{10}}{9}$$

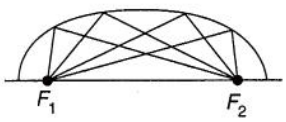
then

$$\frac{x^2}{\frac{10}{9}} + \frac{y^{10}}{10} = 1$$

is the standard form of an ellipse. The intercepts are

$$\left(\frac{\pm\sqrt{10}}{3}, 0\right) \quad \text{and} \quad (0, \pm\sqrt{10})$$

## Focus on Whispering Galleries




The domed roof in the accompanying figure has the shape of an ellipse that has been rotated about its major axis. It can be shown, using basic laws of physics, that a sound uttered at one focus is reflected to the other focus, where it is clearly heard. This property of such rooms is known as the "whispering gallery effect."

Famous whispering galleries include the dome of St. Paul's Cathedral, London; St. John Lateran, Rome; the Salle des Carlatides in the Louvre, Paris; and the original House of Representatives (now the National Statuary Hall in the United States Capitol), Washington, D.C.

**✔ Progress Check**

Write the equation of each ellipse in standard form and determine the intercepts.

a. $2x^2 + 3y^2 = 6$   b. $3x^2 + y^2 = 5$

**Answers**

a. $\frac{x^2}{3} + \frac{y^2}{2} = 1$; $(\pm\sqrt{3}, 0)$, $(0, \pm\sqrt{2})$

b. $\frac{x^2}{\frac{5}{3}} + \frac{y^2}{5} = 1$; $\left(\frac{\pm\sqrt{15}}{3}, 0\right)$, $(0, \pm\sqrt{5})$

## The Hyperbola

The geometric definition of a hyperbola is as follows:

### Definition of a Hyperbola

A **hyperbola** is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a positive constant. Each fixed point is called a **focus** of the hyperbola.

The equation of the hyperbola is in standard form if the two fixed points are on either the $x$-axis or the $y$-axis, and they are equidistant from the origin. In Figure 28(a), we have placed these foci, $F_1$ and $F_2$, on the $x$-axis, and in Figure

28(b), on the $y$-axis. The line through the foci is called the **transverse axis,** and the midpoint of this axis is called the **center** of the hyperbola. The line through the center and perpendicular to the transverse axis is called the **conjugate axis.** The points at which the hyperbola intersects the transverse axis are called the **vertices** ($V_1$ and $V_2$ in Figures 28(a) and 28(b)) of the hyperbola. The two separate parts of the hyperbola are called its **branches.**

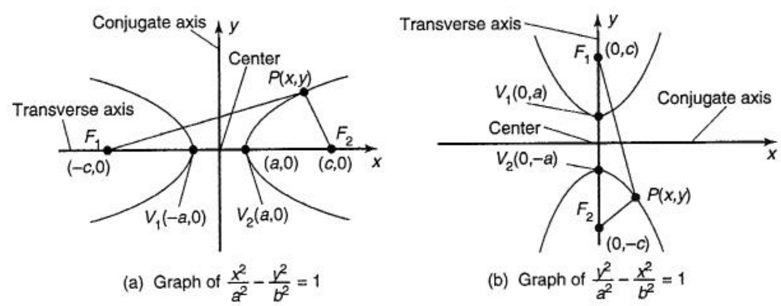



**FIGURE 28** Deriving the Equation of a Hyperbola

Consider placing the point $P$ at either $V_1$ or $V_2$. If the distance $\overline{PF_1} > \overline{PF_2}$ (see Figure 28(a)), we observe (Why?) that

$$\overline{PF_1} - \overline{PF_2} = 2a$$

Alternatively, if $\overline{PF_1} < \overline{PF_2}$ (see Figure 28(b)), we observe (Why?) that

$$\overline{PF_2} - \overline{PF_1} = 2a$$

Therefore

$$|\overline{PF_2} - \overline{PF_1}| = 2a$$

is true for all points $P$ on the hyperbola. The equations of hyperbolas in standard form can be shown (see Exercise 36) to be as follows:

## Standard Forms of the Equation of a Hyperbola with Center (0, 0)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{foci on the } x\text{-axis} \qquad (1)$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{foci on the } y\text{-axis} \qquad (2)$$

These equations indicate that the graphs are symmetric with respect to the $x$-axis, the $y$-axis, and the origin.

Letting $y = 0$, we see that the $x$-intercepts of the graph of Equation (1) are $x = \pm a$. Letting $x = 0$, we find there are no $y$-intercepts since the equation $y^2 = -b^2$ has no real roots. (See Figure 28(a).) Similarly, the graph of Equation (2) has $y$-intercepts of $\pm a$ and no $x$-intercepts. (See Figure 28(b).)

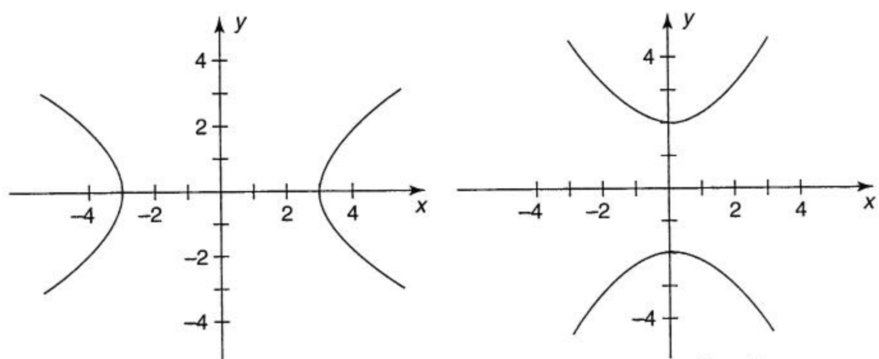
## EXAMPLE 3 USING THE EQUATION OF A HYPERBOLA

Find the intercepts and sketch the graph of each equation.

a. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ b. $\frac{y^2}{4} - \frac{x^2}{3} = 1$

## SOLUTION

a. When $y = 0$, we have $x^2 = 9$ or $x = \pm 3$. The $x$-intercepts are $(3, 0)$ and $(-3, 0)$. With the assistance of a few plotted points, we can sketch the graph as shown in Figure 29(a).




(a) Graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ (b) Graph of $\frac{y^2}{4} - \frac{x^2}{3} = 1$

**FIGURE 29** Graphs for Example 3

b. When $x = 0$, we have $y^2 = 4$ or $y = \pm 2$. The $y$-intercepts are $(0, 2)$ and $(0, -2)$. Plotting a few points, we can sketch the graph as shown in Figure 29(b).

## EXAMPLE 4 USING THE EQUATION OF A HYPERBOLA

Write the equation of the hyperbola $9x^2 - 5y^2 = 10$ in standard form and determine the intercepts.

## SOLUTION

Dividing by 10, we have

$$\frac{9x^2}{10} - \frac{y^2}{2} = 1$$

Rewriting the equation in standard form, we have

$$\frac{x^2}{\frac{10}{9}} + \frac{y^2}{2} = 1$$

The $x$-intercepts are $\left(\frac{\pm\sqrt{10}}{3}, 0\right)$; there are no $y$-intercepts.

**✔ Progress Check**

Write the equation of the hyperbola in standard form and determine the intercepts.

a. $2x^2 - 5y^2 = 6$ b. $4y^2 - x^2 = 5$

**Answers**

a. $\frac{x^2}{3} - \frac{y^2}{\frac{6}{5}} = 1$; $(\pm\sqrt{3}, 0)$ b. $\frac{y^2}{\frac{5}{4}} - \frac{x^2}{5} = 1$; $\left(0, \frac{\pm\sqrt{5}}{2}\right)$

## Asymptotes of a Hyperbola

There is a feature of a hyperbola that enables us to sketch its branches without having to plot many points. By examining the corresponding values of $y$ for values of $x$ that are very far from the origin, we can show that a hyperbola has two asymptotes. (There is a discussion concerning asymptotes in Section 5.1.) Solving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for $y$, we obtain

$$\begin{aligned} y &= \pm\frac{b}{a}\sqrt{x^2 - a^2} \\ &= \pm\frac{b}{a}\sqrt{x^2\left(1 - \frac{a^2}{x^2}\right)} \\ &= \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} \end{aligned}$$

For large values of $|x|$, the term $\frac{a^2}{x^2}$ is very close to 0, in which case $y$ approaches the asymptotes

$$y = \pm\frac{b}{a}x$$

The graph of this hyperbola with its asymptotes is shown in Figure 30. Note that the rectangle with vertices $(\pm a, \pm b)$ is such that the lines that contain the diagonals are the asymptotes.

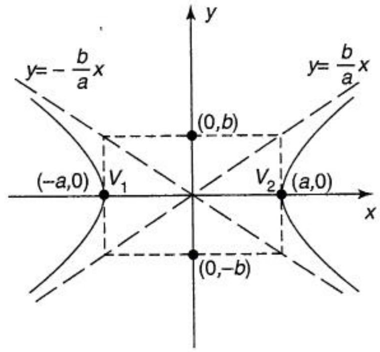



**FIGURE 30** Graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with its Asymptotes

A similar argument may be used with the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

to obtain a different pair of asymptotes. We summarize the results.

## Asymptotes of a Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{has asymptotes } y = \pm\frac{b}{a}x$$

containing the diagonals of the rectangle with vertices $(\pm a, \pm b)$.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{has asymptotes } y = \pm\frac{a}{b}x$$

containing the diagonals of the rectangle with vertices $(\pm b, \pm a)$.

## EXAMPLE 5 ANALYZING THE EQUATION OF A HYPERBOLA

Using asymptotes, sketch the graph of the equation

$$25x^2 - 4y^2 = 100$$

### SOLUTION

We first write the standard form of the equation by dividing both sides by 100, obtaining

$$\frac{x^2}{4} - \frac{y^2}{25} = 1$$

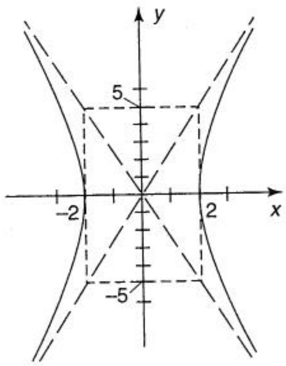



**FIGURE 31** Graph of $25x^2 - 4y^2 = 100$ with its Asymptotes

The points (±2, ±5) form the vertices of the rectangle. Using the fact that (±2, 0) are intercepts, we sketch the graph opening from these points and approaching the asymptotes, as shown in Figure 31.

## EXAMPLE 6 ANALYZING THE EQUATION OF A HYPERBOLA

Using asymptotes, sketch the graph of the equation

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

### SOLUTION

The points (±3, ±2) form the vertices of the rectangle. Since the intercepts are (0, ±2), we sketch the graph opening from these points and approaching the asymptotes, as shown in Figure 32.

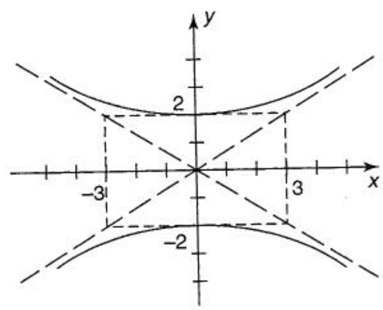



**FIGURE 32** Graph of $\frac{y^2}{4} - \frac{x^2}{9} = 1$ with its Asymptotes

**✔ Progress Check**

Using asymptotes, sketch the graph of the equation

$$4x^2 - 9y^2 = 144$$

**Answers**

See Figure 33.

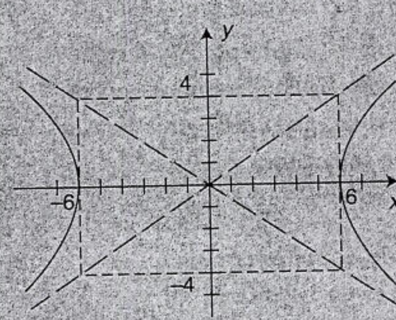



**FIGURE 33** Graph and Asymptotes for Progress Check

## Graphing Calculator Power User's Corner

*Graphing Ellipses and Hyperbolas*

To display an ellipse or hyperbola on your graphing calculator, solve the standard form of the equation for $y$ and then graph the "top" and "bottom" halves in one viewing rectangle. For example, we complete the following to graph:

a. $\frac{x^2}{4} + \frac{y^2}{25} = 1$

GRAPH $y = \frac{\sqrt{100 - 25x^2}}{2}$ and $y = \frac{-\sqrt{100 - 25x^2}}{2}$ in a viewing rectangle that is two times the EQUAL viewing rectangle.

b. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

GRAPH $y = \frac{\sqrt{4x^2 - 36}}{3}$ and $y = \frac{-\sqrt{4x^2 - 36}}{3}$ in the EQUAL viewing rectangle.

c. $\frac{y^2}{4} - \frac{x^2}{3} = 1$

GRAPH $y = \frac{\sqrt{12 + 4x^2}}{\sqrt{3}}$ and $y = \frac{-\sqrt{12 + 4x^2}}{\sqrt{3}}$ in the EQUAL viewing rectangle.

For these graphs, it is worthwhile to use WINDOW values that are multiples of the EQUAL viewing rectangle. They will produce ellipses and hyperbolas in better proportion.

Software is available that will give your graphing calculator the ability to graph conic sections.

## Exercise Set 5.4

In Exercises 1–6, find the intercepts and sketch the graph of the ellipse. Then, determine appropriate WINDOW values, and check your answer using your graphing calculator.

1. $\frac{x^2}{25} + \frac{y^2}{4} = 1$
2. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
3. $\frac{x^2}{8} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{12} + \frac{y^2}{18} = 1$
5. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
6. $\frac{x^2}{1} + \frac{y^2}{3} = 1$

In Exercises 7–16, write the equation of the ellipse in standard form and determine the intercepts. Then, determine appropriate WINDOW values, and GRAPH the equation using your graphing calculator.

7. $4x^2 + 9y^2 = 36$
8. $16x^2 + 9y^2 = 144$
9. $4x^2 + 16y^2 = 16$
10. $25x^2 + 4y^2 = 100$
11. $4x^2 + 16y^2 = 4$
12. $8x^2 + 4y^2 = 32$

13. $8x^2 + 6y^2 = 24$  14. $5x^2 + 6y^2 = 50$

15. $36x^2 + 8y^2 = 9$  16. $5x^2 + 4y^2 = 45$

In Exercises 17–22, find the intercepts and sketch the graph of the hyperbola. Then, determine appropriate WINDOW values, and check your answer using your graphing calculator.

17. $\frac{x^2}{25} - \frac{y^2}{16} = -1$  18. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

19. $\frac{x^2}{36} - \frac{y^2}{1} = 1$  20. $\frac{y^2}{49} - \frac{x^2}{25} = 1$

21. $\frac{x^2}{6} - \frac{y^2}{8} = 1$  22. $\frac{y^2}{8} - \frac{x^2}{10} = -1$

In Exercises 23–28, write the equation of the hyperbola in standard form and determine the intercepts. Then, determine appropriate WINDOW values, and GRAPH the equation using your graphing calculator.

23. $16x^2 - y^2 = 64$  24. $4x^2 - 25y^2 = 100$

25. $4y^2 - 4x^2 = 1$  26. $2x^2 - 3y^2 = 6$

27. $4x^2 - 5y^2 = 20$  28. $25y^2 - 16x^2 = 400$

In Exercises 29–34, use the asymptotes and intercepts to sketch the graph of the hyperbola. Then, determine appropriate WINDOW values, and check your answer using your graphing calculator.

29. $16x^2 - 9y^2 = 144$  30. $16y^2 - 25x^2 = 400$

31. $y^2 - x^2 = 9$  32. $25x^2 - 9y^2 = 225$

33. $\frac{x^2}{25} - \frac{y^2}{36} = 1$  34. $y^2 - 4x^2 = 4$

35. Derive the standard form of the equation of an ellipse from the geometric definition of an ellipse. (*Hint:* In Figure 25, let $P(x, y)$ be any point on the ellipse and let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci. Note that the point $V_2(a, 0)$ lies on the ellipse and that $\overline{V_2F_1} + \overline{V_2F_2} = 2a$. Thus, the sum of the distances $\overline{PF_1} + \overline{PF_2}$ must also equal $2a$. Use the distance formula, simplify, and substitute $b^2 = a^2 - c^2$.)

36. Derive the standard form of the equation of a hyperbola from the geometric definition of a hyperbola. (*Hint:* Proceed in a manner similar to that of Exercise 35 and refer to Figure 28.)

37. An architect decides to install an elliptical window in the sun room of a house. The window is to be 36 inches long and 9 inches high. Find the vertical height of the window 5 inches horizontally from the window's center.

38. Find the distance from the center of the window from Exercise 37 to the place where the window is 20 inches long.

In Exercises 39 and 40, a footbridge is to be constructed over a road. The design of the footbridge is the upper half of an ellipse. Suppose the footbridge is 350 feet long and 40 feet high.

39. Determine the height of the bridge 125 feet from its center.

40. Determine the distance from the center of the bridge at which it is 30 feet high.

41. Find the equation of the ellipse, center at the origin, vertex at (3, 0) and passing through $(\sqrt{5}, \frac{4}{3})$.

42. Find the equation of the ellipse, center at the origin, vertex at (4, 0), and passing through $(\frac{16}{5}, -3)$.

43. Modify Exercise 35 and find the equation of the ellipse, center at the origin, focus at (0, −4), and vertex at (0, 5).

44. Using Exercise 35, find the equation of the ellipse, center at the origin, focus at (2, 0), and vertex at (3, 0).

45. Find the equation of the hyperbola, center at the origin, vertex at (3, 0), and passing through $(5, \frac{4}{3})$.

46. Find the equation of the hyperbola, center at the origin, vertex at (0, 4), and with asymptotes $y = \pm\frac{4}{3}x$.

## 5.5 Translation of Axes

In Figure 34, two sets of coordinate axes are displayed. There is the standard $x$- and $y$-coordinate axes with origin $O$; and there is another set of coordinate axes, $x'$ and $y'$, that are parallel to the $x$-axis and $y$-axis, respectively.

Consider shifting the $x$-axis vertically to the $x'$-axis and shifting the $y$-axis horizontally to the $y'$-axis. This process is called **translation of axes,** and we say that the $x$- and $y$-axes have been translated.

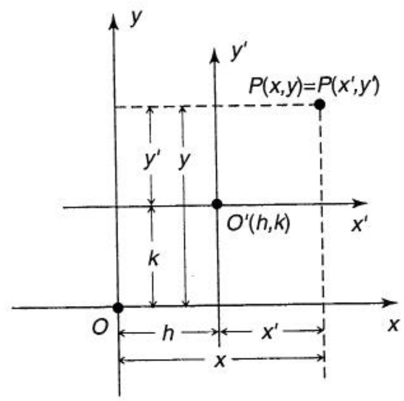



**FIGURE 34** Deriving the Formulas for Translation of Axes

A point $P$ in the plane has coordinates $(x, y)$ with respect to the $xy$-coordinate system and $(x', y')$ with respect to the $x'y'$-coordinate system. Suppose that $O'$, the origin of the $x'y'$ system, has coordinates $(h, k)$ in the $xy$ system. We can obtain the following formulas relating $x$ to $x'$ and $y$ to $y'$.

$$x' = x - h \quad \text{and} \quad y' = y - k$$

### EXAMPLE 1 USING THE FORMULAS FOR TRANSLATION OF AXES

The origin $O'$ of the $x'y'$-coordinate system is at $(-2, 4)$.

a. Express $x'$ and $y'$ in terms of $x$ and $y$, respectively.

b. Find the $x'y'$-coordinates of the point $P$ whose xy-coordinates are $(4, -6)$.

### SOLUTION

a. Substituting $h = -2$ and $k = 4$, we obtain the translation formulas

$$x' = x + 2 \qquad y' = y - 4$$

b. Substituting $x = 4$ and $y = -6$ yields

$$x' = 4 + 2 = 6 \qquad y' = -6 - 4 = -10$$

The $x'y'$-coordinates of $P$ are $(6, -10)$.

**✔ Progress Check**

The origin $O'$ of the $x'y'$-coordinate system is at $(-1, -2)$.

a. Express $x$ and $y$ in terms of $x'$ and $y'$, respectively.

b. Find the $xy$-coordinates of the point $P$ whose $x'y'$-coordinates are $(3, -3)$.

**Answers**

a. $x = x' - 1, y = y' - 2$ b. $(2, -5)$

We can use the translation formulas to transform equations given in the $xy$-coordinate system to equations in the $x'y'$-coordinate system, and vice versa. For example, in the $x'y'$ system, the equation of the circle with center at $O'$ and radius $r$ is

$$(x')^2 + (y')^2 = r^2$$

Substituting the translation formulas

$$x' = x - h \quad \text{and} \quad y' = y - k$$

we find that the equation of this circle in $xy$-coordinates is

$$(x - h)^2 + (y - k)^2 = r^2$$

which is precisely the standard form of the equation of the circle with center at $(h, k)$ and radius $r$, as discussed in Section 5.2.

### EXAMPLE 2 ANALYZING AND SKETCHING A CONIC SECTION

Discuss and sketch the graph of the equation

$$x^2 - 4x + y^2 + 2y + 1 = 0$$

### SOLUTION

We group the terms in $x$ and the terms in $y$

$$(x^2 - 4x \qquad) + (y^2 + 2y \qquad) = -1$$

Completing the square in each variable

$$(x^2 - 4x + 4) + (y^2 + 2y + 1) = -1 + 4 + 1 = 4$$

(Note that the equation is balanced by adding $4 + 1$ to the right-hand side.) We then have

$$(x - 2)^2 + (y + 1)^2 = 4$$

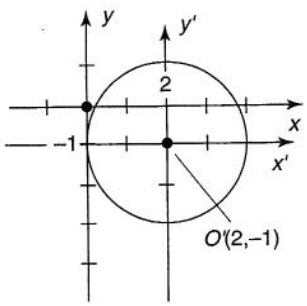



**FIGURE 35** Graph of $x^2 - 4x + y^2 + 2y + 1 = 0$

which is the equation of a circle with center at $(2, -1)$ and radius 2. In terms of the $x'y'$ coordinate system with origin $O'$ at $(2, -1)$, the equation becomes

$$(x')^2+(y')^2 = 4$$

We see that the equation and analysis are simplified by translating the axes to the point $(h, k) = (2, -1)$ as shown in Figure 35.

The technique of translation of axes can be applied to each of the conic sections. The results can be summarized as shown in Table 2. If we write the equation of a conic section in standard form, we can perform a translation of axes to the origin $O'(h, k)$ and then analyze and sketch the graph in the simplified form relative to the $x'y'$-coordinate system.

## EXAMPLE 3 ANALYZING AND SKETCHING A CONIC SECTION

Sketch the graph of the equation

$$x^2 + 4x - 4y^2 + 24y - 48 = 0$$

### SOLUTION

Write the equation as

$$(x^2 + 4x \qquad) - 4(y^2 - 6y \qquad) = 48$$

and complete the square in both $x$ and y.

$$(x^2 + 4x + 4) - 4(y^2 - 6y + 9) = 48 + 4 - 36$$
$$(x + 2)^2 - 4(y - 3)^2 = 16$$

Letting

$$x' = x + 2 \quad \text{and} \quad y' = y - 3$$

we have

$$(x')^2 - 4(y')^2 = 16$$

and divide through by 16 to obtain

$$\frac{(x')^2}{16} - \frac{(y')^2}{4} = 1$$

On the $x'y'$-coordinate system this is seen to be the equation of a hyperbola with center at $O'(-2, 3)$, $a = 4$, $b = 2$. Using the intercepts and asymptotes, the graph is sketched in Figure 36.

## TABLE 2 Standard Forms of Conic Sections

| Comic Section | Standard Form | Characteristics | Example |
|---|---|---|---|
| Circle | $(x-h)^2+(y-k)^2=r^2$ | Center: $(h, k)$<br>Radius: $r$ | $(x-2)^2+(y+4)^2=25$<br>Center: $(2, -4)$<br>Radius: 5 |
| Parabola | $(x-h)^2=4p(y-k)$ | Vertex: $(h, k)$<br>Axis: $x=h$<br>Directrix: $y=k-p$<br>$p>0$: Opens up<br>$p<0$: Opens down | $(x+1)^2=2(y-3)$<br>Vertex: $(-1, 3)$<br>Axis: $x=-1$<br>Directrix: $y=\frac{5}{2}$<br>Opens up |
| | or | | |
| | $(y-k)^2=4p(x-h)$ | Vertex: $(h, k)$<br>Axis: $y=k$<br>Directrix: $x=h-p$<br>$p>0$: Opens right<br>$p<0$: Opens left | $(y+4)^2=-4(x+5)$<br>Vertex: $(-5, -4)$<br>Axis: $y=-4$<br>Directrix: $x=-4$<br>Opens left |
| Ellipse | $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ | Center: $(h, k)$<br>Endpoints of axes of ellipse:<br>$(h \pm a, k)$, $(h, k \pm b)$ | $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$<br>Center: $(1, -2)$<br>Endpoints of axes of ellipse:<br>$(3, -2)$, $(-1, -2)$, $(1, 1)$, $(1, -5)$ |
| Hyperbola | $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ | Center: $(h, k)$<br>Vertices: $(h \pm a, k)$<br>Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$<br>Opens left and right | $\frac{(x+3)^2}{25}-\frac{(y-2)^2}{16}=1$<br>Center: $(-3, 2)$<br>Vertices: $(2, 2)$, $(-8, 2)$<br>Asymptotes: $y-2=\pm\frac{4}{5}(x+3)$<br>Opens left and right |
| | or | | |
| | $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ | Center: $(h, k)$<br>Vertices: $(h, k \pm a)$<br>Asymptotes: $y-k=\pm\frac{a}{b}(x-h)$<br>Opens up and down | $\frac{(y-5)^2}{4}-\frac{(x+1)^2}{25}=1$<br>Center: $(-1, 5)$<br>Vertices: $(-1, 7)$, $(-1, 3)$<br>Asymptotes: $y-5=\pm\frac{2}{5}(x+1)$<br>Opens up and down |

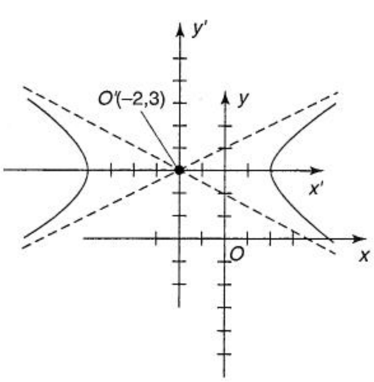



**FIGURE 36** Graph of $x^2+4x-4y^2+24y-48=0$

**WARNING**

To complete the square in the equation

$$2(x^2 + 2x \quad ) - 3(y^2 - 4y \quad ) = 16$$

we must add 1 to the terms in $x$ and 4 to the terms in $y$:

$$2(x^2 + 2x + 1) - 3(y^2 - 4y + 4) = 16 + 2 - 12$$

Note that adding 1 in the first parenthesis results in adding 2 to the left-hand side and is balanced by adding 2 to the right-hand side. Similarly, adding 4 in the second parenthesis results in adding $-12$ to the left-hand side, and this is also balanced on the right-hand side.

**✔ Progress Check**

a. Show that the graph of the equation

$$4x^2 + 16x + y^2 + 2y + 13 = 0$$

is an ellipse.

b. Show that the graph of the equation

$$4y^2 - 24y - 25x^2 - 50x - 89 = 0$$

is a hyperbola.

### EXAMPLE 4 ANALYZING AND SKETCHING A CONIC SECTION

Sketch the graph of the equation

$$x^2 - 4x - 4y - 4 = 0$$

**SOLUTION**

Write the equation as

$$(x^2 - 4x \quad ) = 4y + 4$$

and complete the square in $x$.

$$(x^2 - 4x + 4) = 4y + 4 + 4$$
$$(x - 2)^2 = 4y + 8 = 4(y + 2)$$

Letting

$$x' = x - 2 \quad \text{and} \quad y' = y + 2$$

we obtain

$$(x')^2 = 4y'$$

In the $x'y'$-coordinate system, this is seen to be the equation of a parabola with vertex at $O'(2, -2)$ and $p = 1$. The graph is sketched in Figure 37.