TABLE 11 Properties of Powers $a^n = b$ and Roots $a = b^{1/n}$ for Integer $n > 0$

Example		Property
$2^3 = 8$	$(-2)^3 = -8$	Any power of a real number is a real number.
$8^{1/3} = 2$	$(-8)^{1/3} = -2$	The odd root of a real number is a real number.
$0^n = 0$	$0^{1/n} = 0$	A positive power or root of zero is zero.
$4^2 = 16$	$(-4)^2 = 16$	A positive number raised to an even power equals the negative of that number raised to the same even power.
$(16)^{1/2} = 4$		The principal root of a positive number is a positive number.
$(-4)^{1/2}$ is undefined in the real number system.		The even root of a negative number is not a real number.

EXAMPLE 1 ROOTS OF A REAL NUMBER

Evaluate.

a. $144^{1/2}$

b. $(-8)^{1/3}$

c. $(-25)^{1/2}$

d. $-\left(\frac{1}{16}\right)^{1/4}$

SOLUTION

a. $144^{1/2} = 12$

b. $(-8)^{1/3} = -2$

c. $(-25)^{1/2}$ is not a real number

d. $-\left(\frac{1}{16}\right)^{1/4} = -\frac{1}{2}$

Rational Exponents

Now we are prepared to define $b^{m/n}$. Where m is an integer (positive or negative), n is a positive integer, and $b > 0$ when n is even. We want the rules for exponents to hold for rational exponents as well. That is, we want to have

$$4^{3/2} = 4^{(1/2)(3)} = (4^{1/2})^3 = 2^3 = 8$$

and

$$4^{3/2} = 4^{(3)(1/2)} = (4^3)^{1/2} = (64)^{1/2} = 8$$

To achieve this consistency, we define $b^{m/n}$ for an integer m, a natural number n, and a real number b, by

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}$$

where b must be positive when n is even. With this definition, all the rules of exponents continue to hold when the exponents are rational numbers.

EXAMPLE 2 OPERATIONS WITH RATIONAL EXPONENTS

Simplify.

a. $(-8)^{4/3}$ b. $x^{1/2} \cdot x^{3/4}$ c. $(x^{3/4})^2$ d. $(3x^{2/3}y^{-5/3})^3$

SOLUTION

a. $(-8)^{4/3} = [(-8)^{1/3}]^4 = (-2)^4 = 16$

b. $x^{1/2} \cdot x^{3/4} = x^{1/2+3/4} = x^{5/4}$

c. $(x^{3/4})^2 = x^{(3/4)(2)} = x^{3/2}$

d. $(3x^{2/3}y^{-5/3})^3 = 3^3 \cdot x^{(2/3)(3)}y^{(-5/3)(3)} = 27x^2y^{-5} = \dfrac{27x^2}{y^5}$

Focus: When Is a Proof Not a Proof?

Books of mathematical puzzles love to include "proofs" that lead to false or contradictory results. Of course, there is always an incorrect step hidden somewhere in the proof. The error may be subtle, but a good grounding in the fundamentals of mathematics will enable you to catch it.

Examine the following "proof."

$$1 = 1^{1/2} \quad (1)$$

$$= [(-1)^2]^{1/2} \quad (2)$$

$$= (-1)^{2/2} \quad (3)$$

$$= (-1)^1 \quad (4)$$

$$= -1 \quad (5)$$

The result is obviously contradictory: we can't have $1 = -1$. Yet each step seems to be legitimate. Did you spot the flaw? The rule

$$(b^m)^{1/n} = b^{m/n}$$

used in going from Equation (2) to (3) does not apply when n is even and b is negative.

✔ Progress Check

Simplify. Assume all variables are positive real numbers.

a. $27^{4/3}$ b. $(a^{1/2}b^{-2})^{-2}$ c. $\left(\frac{x^{1/3}y^{2/3}}{z^{5/6}}\right)^{12}$

Answers

a. 81 b. $\frac{b^4}{a}$ c. $\frac{x^4y^8}{z^{10}}$

Radicals

The symbol $\sqrt{b}$ is an alternative way of writing $b^{1/2}$; that is, $\sqrt{b}$ denotes the nonnegative square root of b. The symbol $\sqrt{\ }$ is called a **radical sign**, and $\sqrt{b}$ is called the **principal square root** of b. Thus,

$$\sqrt{25} = 5 \qquad \sqrt{0} = 0 \qquad \sqrt{-25} \text{ is undefined}$$

In general, the symbol $\sqrt[n]{b}$ is an alternative way of writing $b^{1/n}$, the principal nth root of b. Of course, we must apply the same restrictions to $\sqrt[n]{b}$ that we established for $b^{1/n}$. In summary:

$$\sqrt[n]{b} = b^{1/n} = a \qquad \text{where } a^n = b$$

with these restrictions:

- if n is even and $b < 0$, $\sqrt[n]{b}$ is not a real number;
- if n is even and $b \geq 0$, $\sqrt[n]{b}$ is the *nonnegative* number a satisfying $a^n = b$.

WARNING

Many students are accustomed to writing $\sqrt{4} = \pm 2$. This is incorrect since the symbol $\sqrt{\ }$ indicates the *principal* square root, which is nonnegative. Get in the habit of writing $\sqrt{4} = 2$. If you want to indicate *all* square roots of 4, write $\pm\sqrt{4} = \pm 2$.

In short, $\sqrt[n]{b}$ is the **radical form of** $b^{1/n}$. We can switch back and forth from one form to the other. For instance,

$$\sqrt[3]{7} = 7^{1/3} \qquad (11)^{1/5} = \sqrt[5]{11}$$

Finally, we treat the radical form of $b^{m/n}$ where m is an integer and n is a positive integer as follows:

$$b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m}$$

and

$$b^{m/n} = (b^{1/n})^m = (\sqrt[n]{b})^m$$

Thus

$$8^{2/3} = (8^2)^{1/3} = \sqrt[3]{8^2}$$
$$= (8^{1/3})^2 = (\sqrt[3]{8})^2$$

(Check that the last two expressions have the same value.)

EXAMPLE 3 RADICALS AND RATIONAL EXPONENTS

Change from radical form to rational exponent form or vice versa. Assume all variables are nonzero.

a. $(2x)^{-3/2}, \quad x > 0$

b. $\dfrac{1}{\sqrt[7]{y^4}}$

c. $(-3a)^{3/7}$

d. $\sqrt{x^2 + y^2}$

SOLUTION

a. $(2x)^{-3/2} = \dfrac{1}{(2x)^{3/2}} = \dfrac{1}{\sqrt{8x^3}}$

b. $\dfrac{1}{\sqrt[7]{y^4}} = \dfrac{1}{y^{4/7}} = y^{-4/7}$

c. $(-3a)^{3/7} = \sqrt[7]{-27a^3}$

d. $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

✔ Progress Check

Change from radical form to rational exponent form or vice versa. Assume all variables are positive real numbers.

a. $\sqrt[4]{2rs^3}$ b. $(x + y)^{5/2}$ c. $y^{-5/4}$ d. $\dfrac{1}{\sqrt[4]{m^5}}$

Answers

a. $(2r)^{1/4}s^{3/4}$ b. $\sqrt{(x + y)^5}$ c. $\dfrac{1}{\sqrt[4]{y^5}}$ d. $m^{-5/4}$

Since radicals are just another way of writing exponents, the properties of radicals can be derived from the properties of exponents. In Table 12, n is a positive integer, a and b are real numbers, and all radicals are real numbers.

TABLE 12 Properties of Radicals

Example	Property
$\sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 4$	$\sqrt[n]{b^m} = (\sqrt[n]{b})^m$
$\sqrt{4}\sqrt{9} = \sqrt{36} = 6$	$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
$\dfrac{\sqrt[3]{8}}{\sqrt[3]{27}} = \sqrt[3]{\dfrac{8}{37}} = \dfrac{2}{3}$	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
$\sqrt[3]{(-2)^3} = -2$	$\sqrt[n]{a^n} = a$ if n is odd
$\sqrt{(-2)^2} = \lvert -2 \rvert = 2$	$\sqrt[n]{a^n} = \lvert a \rvert$ if n is even

Here are some examples using these properties.

EXAMPLE 4 OPERATIONS WITH RADICALS

Simplify.

a. $\sqrt{18}$ b. $\sqrt[3]{-54}$ c. $2\sqrt[3]{8x^3y}$ d. $\sqrt{x^6}$

SOLUTION

a. $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

b. $\sqrt[3]{-54} = \sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$

c. $2\sqrt[3]{8x^3y} = 2\sqrt[3]{8}\sqrt[3]{x^3}\sqrt[3]{y} = 2(2)(x)\sqrt[3]{y} = 4x\sqrt[3]{y}$

d. $\sqrt{x^6} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} = \lvert x \rvert \cdot \lvert x \rvert \cdot \lvert x \rvert = \lvert x \rvert^3$

WARNING

The properties of radicals state that

$$\sqrt{x^2} = \lvert x \rvert$$

It is a common error to write $\sqrt{x^2} = x$. This can lead to the conclusion that $\sqrt{(-6)^2} = -6$. Since the symbol $\sqrt{}$ represents the principal, or nonnegative, square root of a number, the result cannot be negative. It is therefore essential to write $\sqrt{x^2} = \lvert x \rvert$ (and, in fact, $\sqrt[n]{x^n} = \lvert x \rvert$ whenever n is even) unless we know that $x \geq 0$, in which case we can write $\sqrt{x^2} = x$.

Simplifying Radicals

A radical is said to be in **simplified form** when the following conditions are satisfied:

1. $\sqrt[n]{b^m}$ has $m < n$;

2. $\sqrt[n]{b^m}$ has no common factors between m and n;

3. A denominator is free of radicals.

The first two conditions can always be met by using the properties of radicals and by writing radicals in exponent form. For example,

$$\sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = \sqrt[3]{x^3}\sqrt[3]{x} = x\sqrt[3]{x}$$

and

$$\sqrt[6]{x^4} = x^{4/6} = x^{2/3} = \sqrt[3]{x^2}$$

The third condition can always be satisfied by multiplying the fraction by a properly chosen form of unity, a process called **rationalizing the denominator.** For example, to rationalize $\frac{1}{\sqrt{3}}$ we proceed as follows:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{3}$$

In this connection, a useful formula is

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

which we will apply in the following examples.

EXAMPLE 5 RATIONALIZING DENOMINATORS

Rationalize the denominator. Assume all variables denote positive numbers.

a. $\sqrt{\frac{x}{y}}$ b. $\frac{4}{\sqrt{5} - \sqrt{2}}$ c. $\frac{5}{\sqrt{x} + 2}$ d. $\frac{5}{\sqrt{x + 2}}$

SOLUTION

a. $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}} = \frac{\sqrt{x}}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{\sqrt{xy}}{\sqrt{y^2}} = \frac{\sqrt{xy}}{y}$

b. $\frac{4}{\sqrt{5}-\sqrt{2}} = \frac{4}{\sqrt{5}-\sqrt{2}} \cdot \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} = \frac{4(\sqrt{5} + \sqrt{2})}{5-2} = \frac{4}{3}(\sqrt{5} + \sqrt{2})$

c. $\frac{5}{\sqrt{x} + 2} = \frac{5}{\sqrt{x} + 2} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2} = \frac{5(\sqrt{x} - 2)}{x - 4}$

d. $\frac{5}{\sqrt{x + 2}} = \frac{5}{\sqrt{x + 2}} \cdot \frac{\sqrt{x + 2}}{\sqrt{x + 2}} = \frac{5\sqrt{x + 2}}{x + 2}$

✔ Progress Check

Rationalize the denominator. Assume all variables denote positive numbers.

a. $\dfrac{-9xy^3}{\sqrt{3xy}}$ b. $\dfrac{-6}{\sqrt{2}+\sqrt{6}}$ c. $\dfrac{4}{\sqrt{x}-\sqrt{y}}$

Answers

a. $-3y^2\sqrt{3xy}$ b. $\dfrac{3}{2}(\sqrt{2}-\sqrt{6})$ c. $\dfrac{4(\sqrt{x}+\sqrt{y})}{x-y}$

There are times in mathematics when it is necessary to rationalize the numerator instead of the denominator. Although this is in opposition to a simplified form, we illustrate this technique with the following example. Note that if an expression does not display a denominator, we assume a denominator of 1.

EXAMPLE 6 RATIONALIZING NUMERATORS

Rationalize the numerator. Assume all variables denote positive numbers.

a. $\dfrac{4}{3}(\sqrt{5}+\sqrt{2})$ b. $\dfrac{x-\sqrt{3}}{x+4}$

c. $\sqrt{x}+4$ d. $\dfrac{\sqrt{x}-2}{x-4}$

SOLUTION

a. $\dfrac{4}{3}(\sqrt{5}+\sqrt{2})\cdot\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}=\dfrac{4}{3}\,\dfrac{(5-2)}{(\sqrt{5}-\sqrt{2})}=\dfrac{4}{\sqrt{5}-\sqrt{2}}$

See Example 5 (b).

b. $\dfrac{x-\sqrt{3}}{x+4}\cdot\dfrac{x+\sqrt{3}}{x+\sqrt{3}}=\dfrac{x^2-3}{(x+4)(x+\sqrt{3})}$

c. $\dfrac{\sqrt{x}+4}{1}\cdot\dfrac{\sqrt{x}-4}{\sqrt{x}-4}=\dfrac{x-16}{\sqrt{x}-4}$

d. $\dfrac{\sqrt{x}-2}{x-4}\cdot\dfrac{\sqrt{x}+2}{\sqrt{x}+2}=\dfrac{x-4}{(x-4)(\sqrt{x}+2)}=\dfrac{1}{\sqrt{x}+2},\qquad x\neq 4$

EXAMPLE 7 SIMPLIFIED FORMS WITH RADICALS

Write in simplified form. Assume all variables denote positive numbers.

a. $\sqrt[4]{y^5}$ b. $\sqrt{\dfrac{8x^3}{y}}$ c. $\sqrt[6]{\dfrac{x^3}{y^2}}$

SOLUTION

a. $\sqrt[4]{y^5} = \sqrt[4]{y^4 \cdot y} = \sqrt[4]{y^4}\sqrt[4]{y} = y\sqrt[4]{y}$

b. $\sqrt{\dfrac{8x^3}{y}} = \dfrac{\sqrt{(4x^2)(2x)}}{\sqrt{y}} = \dfrac{\sqrt{4x^2}\sqrt{2x}}{\sqrt{y}} = \dfrac{2x\sqrt{2x}}{\sqrt{y}} = \dfrac{2x\sqrt{2x}}{\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} = \dfrac{2x\sqrt{2xy}}{y}$

c. $\sqrt[6]{\dfrac{x^3}{y^2}} = \dfrac{\sqrt[6]{x^3}}{\sqrt[6]{y^2}} = \dfrac{\sqrt{x}}{\sqrt[3]{y}} = \dfrac{\sqrt{x}}{\sqrt[3]{y}} \cdot \dfrac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} = \dfrac{\sqrt{x}\sqrt[3]{y^2}}{y}$

✔ Progress Check

Write in simplified form. Assume all variables denote positive numbers.

a. $\sqrt{75}$ b. $\sqrt{\dfrac{18x^6}{y}}$ c. $\sqrt[3]{ab^4c^7}$ d. $\dfrac{-2xy^3}{\sqrt[4]{32x^3y^5}}$

Answers

a. $5\sqrt{3}$ b. $\dfrac{3x^3\sqrt{2y}}{y}$ c. $bc^2\sqrt[3]{abc}$ d. $-\dfrac{y}{2}\sqrt[4]{8xy^3}$

Operations with Radicals

We can add or subtract expressions involving exactly the same radical forms. For example,

$$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$$

since

$$2\sqrt{2} + 3\sqrt{2} = (2 + 3)\sqrt{2} = 5\sqrt{2}$$

and

$$3\sqrt[3]{x^2y} - 7\sqrt[3]{x^2y} = -4\sqrt[3]{x^2y}$$

EXAMPLE 8 ADDITION AND SUBTRACTION OF RADICALS

Write in simplified form. Assume all variables denote positive numbers.

a. $7\sqrt{5} + 4\sqrt{3} - 9\sqrt{5}$

b. $\sqrt[3]{x^2y} - \dfrac{1}{2}\sqrt{xy} - 3\sqrt[3]{x^2y} + 4\sqrt{xy}$

SOLUTION

a. $7\sqrt{5} + 4\sqrt{3} - 9\sqrt{5} = -2\sqrt{5} + 4\sqrt{3}$

b. $\sqrt[3]{x^2y} - \dfrac{1}{2}\sqrt{xy} - 3\sqrt[3]{x^2y} + 4\sqrt{xy} = -2\sqrt[3]{x^2y} + \dfrac{7}{2}\sqrt{xy}$

WARNING

$$\sqrt{9} + \sqrt{16} \neq \sqrt{25}$$

You can perform addition only with identical radical forms. *Adding unlike radicals is one of the most common mistakes made by students in algebra!* You can easily verify that

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

The product of $\sqrt[n]{a}$ and $\sqrt[m]{b}$ can be readily simplified only when $m = n$. Thus,

$$\sqrt[5]{x^2y} \cdot \sqrt[5]{xy} = \sqrt[5]{x^3y^2}$$

but

$$\sqrt[3]{x^2y} \cdot \sqrt[5]{xy}$$

cannot be readily simplified.

EXAMPLE 9 MULTIPLICATION OF RADICALS

Multiply and simplify.

a. $2\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y^2}$ b. $\sqrt[5]{a^2b}\sqrt[5]{ab}\sqrt[5]{ab^2}$

SOLUTION

a. $2\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y^2} = 2\sqrt[3]{x^3y^4} = 2xy\sqrt[3]{y}$

b. $\sqrt[5]{a^2b}\sqrt[5]{ab}\sqrt[5]{ab^2} = \sqrt[5]{a^3b^3}\sqrt{ab}$

Calculator Alert

Most calculators have a $\boxed{\sqrt{\ }}$ key. Scientific and graphing calculators sometimes have a special key to evaluate other roots. This key may be $\boxed{\sqrt[x]{y}}$, $\boxed{\sqrt[x]{\ }}$, or $\boxed{x^{1/y}}$. If your calculator does not have a special root key, you can use the power key to evaluate roots.

Example: Show that $\sqrt[5]{12} = 1.64375183$.

Solution: If your calculator has a root key, evaluate $5\boxed{\sqrt[x]{y}}12$. Otherwise, evaluate $12\boxed{x^y}(1 \div 5)$ or $12\boxed{\wedge}(1 \div 5)$ on your calculator.

Exercise Set 1.7

In Exercises 1–12, simplify, and write the answer using only positive exponents.

1. $16^{3/4}$
2. $(-125)^{-1/3}$
3. $(-64)^{-2/3}$
4. $c^{1/4}c^{-2/3}$
5. $\dfrac{2x^{1/3}}{x^{-3/4}}$
6. $\dfrac{y^{-2/3}}{y^{1/5}}$
7. $\left(\dfrac{x^{3/2}}{x^{2/3}}\right)^{1/6}$
8. $\dfrac{125^{4/3}}{125^{2/3}}$
9. $(x^{1/3}y^2)^6$
10. $(x^6y^4)^{-1/2}$
11. $\left(\dfrac{x^{15}}{y^{10}}\right)^{3/5}$
12. $\left(\dfrac{x^{18}}{y^{-6}}\right)^{2/3}$

In Exercises 13–18, write the expression in radical form.

13. $\left(\dfrac{1}{4}\right)^{2/5}$
14. $x^{2/3}$
15. $a^{3/4}$
16. $(-8x^2)^{2/5}$
17. $(12x^3y^{-2})^{2/3}$
18. $\left(\dfrac{8}{3}x^{-2}y^{-4}\right)^{-3/2}$

In Exercises 19–24, write the expression in exponent form.

19. $\sqrt[4]{8^3}$
20. $\sqrt[5]{3^2}$
21. $\dfrac{1}{\sqrt[5]{(-8)^2}}$
22. $\dfrac{1}{\sqrt[3]{x^7}}$
23. $\dfrac{1}{\sqrt[4]{\frac{4}{9}a^3}}$
24. $\sqrt[5]{(2a^2b^3)^4}$

In Exercises 25–33, evaluate the expression. Verify your answer using your calculator.

25. $\sqrt{\dfrac{4}{9}}$
26. $\sqrt{\dfrac{25}{4}}$
27. $\sqrt[4]{-81}$
28. $\sqrt[3]{\dfrac{1}{27}}$
29. $\sqrt{(5)^2}$
30. $\sqrt{\left(\dfrac{-1}{3}\right)^2}$
31. $\sqrt{\left(\dfrac{5}{4}\right)^2}$
32. $\sqrt{\left(-\dfrac{7}{2}\right)^2}$
33. $(14.43)^{3/2}$

In Exercises 34–36, provide a real value for each variable to demonstrate the result.

34. $\sqrt{x^2} \neq x$
35. $\sqrt{x^2 + y^2} \neq x + y$
36. $\sqrt{x}\sqrt{y} \neq xy$

In Exercises 37–56, write the expression in simplified form. Every variable represents a positive number.

37. $\sqrt{48}$
38. $\sqrt{200}$
39. $\sqrt[3]{54}$
40. $\sqrt{x^8}$
41. $\sqrt[3]{y^7}$
42. $\sqrt[4]{b^{14}}$
43. $\sqrt[4]{96x^{10}}$
44. $\sqrt{x^5y^4}$
45. $\sqrt{x^5y^3}$
46. $\sqrt[3]{24b^{10}c^{14}}$
47. $\sqrt[4]{16x^8y^5}$
48. $\sqrt{20x^5y^7z^4}$
49. $\sqrt{\dfrac{1}{5}}$
50. $\dfrac{4}{3\sqrt{11}}$
51. $\dfrac{1}{\sqrt{3y}}$
52. $\sqrt{\dfrac{2}{y}}$
53. $\dfrac{4x^2}{\sqrt{2x}}$
54. $\dfrac{8a^2b^2}{2\sqrt{2b}}$
55. $\sqrt[3]{x^2y^7}$
56. $\sqrt[4]{48x^8y^6z^2}$

In Exercises 57–66, simplify and combine terms.

57. $2\sqrt{3} + 5\sqrt{3}$
58. $4\sqrt[3]{11} - 6\sqrt[3]{11}$
59. $3\sqrt{x} + 4\sqrt{x}$
60. $3\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$
61. $2\sqrt{27} + \sqrt{12} - \sqrt{48}$
62. $\sqrt{20} - 4\sqrt{45} + \sqrt{80}$
63. $\sqrt[3]{40} + \sqrt{45} - \sqrt[3]{135} + 2\sqrt{80}$
64. $\sqrt{2abc} - 3\sqrt{8abc} + \sqrt{\dfrac{abc}{2}}$
65. $2\sqrt{5} - (3\sqrt{5} + 4\sqrt{5})$
66. $2\sqrt{18} - (3\sqrt{12} - 2\sqrt{75})$

In Exercises 67–74, multiply and simplify.

67. $\sqrt{3}(\sqrt{3} + 4)$
68. $\sqrt{8}(\sqrt{2} - \sqrt{3})$
69. $3\sqrt[3]{x^2y}\sqrt[3]{xy^2}$
70. $-4\sqrt[5]{x^2y^3}\sqrt[5]{x^4y^2}$

71. $(\sqrt{2}-\sqrt{3})^2$

72. $(\sqrt{8}-2\sqrt{2})(\sqrt{2}+2\sqrt{8})$

73. $(\sqrt{3x}+\sqrt{2y})(\sqrt{3x}-2\sqrt{2y})$

74. $(\sqrt[3]{2x}+3)(\sqrt[3]{2x}-3)$

In Exercises 75–86, rationalize the denominator.

75. $\dfrac{3}{\sqrt{2}+3}$ 76. $\dfrac{-3}{\sqrt{7}-9}$

77. $\dfrac{-2}{\sqrt{3}-4}$ 78. $\dfrac{3}{\sqrt{x}-5}$

79. $\dfrac{-3}{3\sqrt{a}+1}$ 80. $\dfrac{4}{2-\sqrt{2y}}$

81. $\dfrac{-3}{5+\sqrt{5y}}$ 82. $\dfrac{\sqrt{3}}{\sqrt{3}-5}$

83. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ 84. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

85. $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ 86. $\dfrac{2\sqrt{a}}{\sqrt{2x}+\sqrt{y}}$

In Exercises 87–90, rationalize the numerator.

87. $\sqrt{12}-\sqrt{10}$ 88. $\dfrac{3-\sqrt{x}}{x-9}$

89. $\dfrac{\sqrt{x}-4}{16-x}$ 90. $\dfrac{2-\sqrt{x+1}}{3-x}$

In Exercises 91 and 92, provide real values for x and y and a positive integer value for n to demonstrate the result.

91. $\sqrt{x}+\sqrt{y}\neq\sqrt{x+y}$ 92. $\sqrt[n]{x^n+y^n}\neq x+y$

93. Find the step in the following "proof" that is incorrect. Explain.

$$2=\sqrt{4}=\sqrt{(-2)(-2)}=\sqrt{-2}\sqrt{-2}=-2$$

94. Prove that $|ab|=|a||b|$. (*Hint:* Begin with $|ab|=\sqrt{(ab)^2}$.)

95. Simplify the following.

a. $\sqrt{x\sqrt{x\sqrt{x}}}$ b. $(x^{1/2}-x^{-1/2})^2$

c. $\sqrt{1+x^2}-\dfrac{\sqrt{1+x^2}}{2}$

d. $\sqrt[5]{\dfrac{3^4+3^4+3^4}{5^4+5^4+5^4+5^4+5^4}}$

e. $\dfrac{5(1+x^2)^{1/2}-5x^2(1+x^2)^{-1/2}}{1+x^2}$

96. Write the following in simplest radical form.

a. $\sqrt{a^{-2}+c^{-2}}$

b. $\sqrt{1-\left(\frac{a}{c}\right)^2}$

c. $\sqrt{x+\dfrac{1}{x}+2}$

97. The frequency of an electrical circuit is given by

$$\frac{1}{2\pi}\sqrt{\frac{Lc_1c_2}{c_1+c_2}}$$

Make the denominator radical free. (*Hint:* Use the techniques for rationalizing the denominator.)

98. Use your calculator to find $\sqrt{0.4}$, $\sqrt{0.04}$, $\sqrt{0.004}$, $\sqrt{0.0004}$, and so on, until you see a pattern. Can you state a rule about the value of

$$\sqrt{\frac{a}{10^n}}$$

where a is a perfect square and n is a positive integer? Under what circumstances does this expression have an integer value? Test your rule for large values of n.

1.8 Complex Numbers

One of the central problems in algebra is to find solutions to a given polynomial equation. This problem will be discussed in later chapters of this book. For now, observe that there is no real number that satisfies a polynomial equation such as

$$x^2 = -4$$

since the square of a real number is always nonnegative.

To resolve this problem, mathematicians created a new number system built upon an **imaginary unit** i, defined by $i = \sqrt{-1}$. This number i has the property that when we square both sides of the equation we have $i^2 = -1$, a result that cannot be obtained with real numbers. By definition,

$$i = \sqrt{-1}$$
$$i^2 = -1$$

We also assume that i behaves according to all the algebraic laws we have already developed (with the exception of the rules for inequalities for real numbers). This allows us to simplify higher powers of i. Thus,

$$i^3 = i^2 \cdot i = (-1)i = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

Now we may simplify i^n when n is any natural number. Since $i^4 = 1$, we seek the highest multiple of 4 that is less than or equal to n. For example,

$$i^5 = i^4 \cdot i = (1) \cdot i = i$$
$$i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = (1)^6 \cdot i^3 = i^3 = -i$$

EXAMPLE 1 IMAGINARY UNIT *i*

Simplify.

a. i^{51} b. $-i^{74}$

SOLUTION

a. $i^{51} = i^{48} \cdot i^3 = (i^4)^{12} \cdot i^3 = (1)^{12} \cdot i^3 = i^3 = -i$

b. $-i^{74} = -i^{72} \cdot i^2 = -(i^4)^{18} \cdot i^2 = -(1)^{18} \cdot i^2 = -(1)(-1) = 1$

We may also write square roots of negative numbers in terms of i. For example,

$$\sqrt{-25} = i\sqrt{25} = 5i$$

and, in general, we define

$$\sqrt{-a} = i\sqrt{a} \qquad \text{for } a > 0$$

Any number of the form bi, where b is a real number, is called an **imaginary number.**

WARNING

$$\sqrt{-4}\sqrt{-9} \neq \sqrt{36}$$

The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ holds only when $a \geq 0$ and $b \geq 0$. Instead, write

$$\sqrt{-4}\sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$$

Having created imaginary numbers, we next combine real and imaginary numbers. We say that $a + bi$ is a **complex number** where a and b are real numbers. The number a is called the **real part** of $a + bi$, and b is called the **imaginary part.** The following are examples of complex numbers.

$$3 + 2i \qquad 2 - i \qquad -2i \qquad \frac{4}{5} + \frac{1}{5}i$$

Note that every real number a can be written as a complex number by choosing $b = 0$. Thus,

$$a = a + 0i$$

We see that the real number system is a subset of the complex number system. The desire to find solutions to every quadratic equation has led mathematicians to create a more comprehensive number system, which incorporates all previous number systems. We will show in a later chapter that complex numbers are all that we need to provide solutions to any polynomial equation.

EXAMPLE 2 COMPLEX NUMBERS $a + bi$

Write as a complex number:

a. $-\frac{1}{2}$ b. $\sqrt{-9}$ c. $-1 - \sqrt{-4}$

SOLUTION

a. $-\frac{1}{2} = -\frac{1}{2} + 0i$

b. $\sqrt{-9} = i\sqrt{9} = 3i = 0 + 3i$

c. $-1 - \sqrt{-4} = -1 - i\sqrt{4} = -1 - 2i$

Do not be concerned by the word "complex." You already have all the basic tools you need to tackle this number system. We will next define operations with complex numbers in such a way that the rules for the real numbers and the imaginary unit i continue to hold. We begin with equality and say that two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal; that is,

$$a + bi = c + di \quad \text{if} \quad a = c \quad \text{and} \quad b = d$$

EXAMPLE 3 EQUALITY OF COMPLEX NUMBERS

Solve the equation $x + 3i = 6 - yi$ for x and y.

SOLUTION

Equating the real parts, we have $x = 6$; equating the imaginary parts, $3 = -y$ or $y = -3$.

Complex numbers are added and subtracted by adding or subtracting the real parts and by adding or subtracting the imaginary parts.

Addition and Subtraction of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Note that the sum or difference of two complex numbers is again a complex number.

EXAMPLE 4 ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

Perform the indicated operations.

a. $(7 - 2i) + (4 - 3i)$ b. $14 - (3 - 8i)$

SOLUTION

a. $(7 - 2i) + (4 - 3i) = (7 + 4) + (-2 - 3)i = 11 - 5i$

b. $14 - (3 - 8i) = (14 - 3) + 8i = 11 + 8i$

✔ Progress Check

Perform the indicated operations.

a. $(-9 + 3i) + (6 - 2i)$ b. $7i - (3 + 9i)$

Answers

a. $-3 + i$ b. $-3 - 2i$

We now define multiplication of complex numbers in a manner that permits the commutative, associative, and distributive laws to hold, along with the definition $i^2 = -1$. We must have

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\\ &= ac + (ad + bc)i + bd(-1)\\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

The rule for multiplication is

Multiplication of Complex Numbers

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

This result demonstrates that the product of two complex numbers is a complex number. It need not be memorized. Use the distributive law to form all the products and the substitution $i^2 = -1$ to simplify.

EXAMPLE 5 MULTIPLICATION OF COMPLEX NUMBERS

Find the product of $(2 - 3i)$ and $(7 + 5i)$.

SOLUTION

$$\begin{aligned}(2 - 3i)(7 + 5i) &= 2(7 + 5i) - 3i(7 + 5i)\\ &= 14 + 10i - 21i - 15i^2\\ &= 14 - 11i - 15(-1)\\ &= 29 - 11i\end{aligned}$$

✔ Progress Check

Find the product.

a. $(-3 - i)(4 - 2i)$ b. $(-4 - 2i)(2 - 3i)$

Answers

a. $-14 + 2i$ b. $-14 + 8i$

The complex number $a - bi$ is called the **complex conjugate**, or simply the **conjugate**, of the complex number $a + bi$. For example, $3 - 2i$ is the conjugate of $3 + 2i$, $4i$ is the conjugate of $-4i$, and 2 is the conjugate of 2. Forming the product $(a + bi)(a - bi)$, we have

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 + b^2 \quad \text{Since } i^2 = -1\end{aligned}$$

Because a and b are real numbers, $a^2 + b^2$ is also a real number. We can summarize this result as follows:

The Complex Conjugate and Multiplication

The complex conjugate of $a + bi$ is $a - bi$. The product of a complex number and its conjugate is a real number.

$$(a + bi)(a - bi) = a^2 + b^2$$

Before we examine the quotient of two complex numbers, we consider the reciprocal of $a + bi$, namely, $\frac{1}{a + bi}$. This may be simplified by multiplying both numerator and denominator by the conjugate of the denominator.

$$\frac{1}{a + bi} = \left(\frac{1}{a + bi}\right)\left(\frac{a - bi}{a - bi}\right) = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

In general, the quotient of two complex numbers

$$\frac{a + bi}{c + di}$$

is simplified in a similar manner, that is, by multiplying both numerator and denominator by the conjugate of the denominator.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

Division of Complex Numbers

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \qquad c^2 + d^2 \neq 0$$

This result demonstrates that the quotient of two complex numbers is a complex number. Instead of memorizing this formula for division, remember that quotients of complex numbers may be simplified by multiplying the numerator and denominator by the conjugate of the denominator.

EXAMPLE 6 DIVISION OF COMPLEX NUMBERS

a. Write the quotient $\frac{-2 + 3i}{3 - 2i}$ in the form $a + bi$.

b. Write the reciprocal of $2 - 5i$ in the form $a + bi$.

SOLUTION

a. Multiplying numerator and denominator by the conjugate of the denominator, $3 + 2i$, we have

$$\frac{-2+3i}{3-2i} = \frac{-2+3i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{-6-4i+9i+6i^2}{3^2+2^2} = \frac{-6+5i+6(-1)}{9+4}$$

$$= \frac{-12+5i}{13} = -\frac{12}{13} + \frac{5}{13}i$$

b. The reciprocal is $\frac{1}{2-5i}$. Multiplying both numerator and denominator by the conjugate $2 + 5i$, we have

$$\frac{1}{2-5i} \cdot \frac{2+5i}{2+5i} = \frac{2+5i}{2^2+5^2} = \frac{2+5i}{29} = \frac{2}{29} + \frac{5}{29}i$$

Verify that

$$(2-5i)\left(\frac{2}{29} + \frac{5}{29}i\right) = 1$$

✔ Progress Check

Write the following in the form $a + bi$.

a. $\frac{4-2i}{5+2i}$ b. $\frac{1}{2-3i}$ c. $\frac{-3i}{3+5i}$

Answers

a. $\frac{16}{29} - \frac{18}{29}i$ b. $\frac{2}{13} + \frac{3}{13}i$ c. $-\frac{15}{34} - \frac{9}{34}i$

Calculator Alert

Some scientific and graphing calculators have the capability to do computations with complex numbers. Consult your owner's manual for details. The owner's manual may be available online. Look up your calculator by model and number.

Exercise Set 1.8

Simplify in Exercises 1–9.

1. i^{60}
2. i^{27}
3. i^{83}
4. $-i^{54}$
5. $-i^{33}$
6. i^{-15}
7. i^{-84}
8. $-i^{39}$
9. $-i^{-25}$

In Exercises 10–21, write the number in the form $a + bi$.

10. 2
11. $-\frac{3}{4}$
12. -0.3
13. $\sqrt{-25}$
14. $-\sqrt{-5}$
15. $-\sqrt{-36}$
16. $-\sqrt{-18}$
17. $3 - \sqrt{-49}$
18. $-\frac{3}{2} - \sqrt{-72}$
19. $0.3 - \sqrt{-98}$
20. $-0.5 + \sqrt{-32}$
21. $-2 - \sqrt{-16}$

In Exercises 22–26, solve for x and y.

22. $(x + 2) + (2y - 1)i = -1 + 5i$
23. $(3x - 1) + (y + 5)i = 1 - 3i$
24. $\left(\frac{1}{2}x + 2\right) + (3y - 2)i = 4 - 7i$
25. $(2y + 1) - (2x - 1)i = -8 + 3i$
26. $(y - 2) + (5x - 3)i = 5$

In Exercises 27–42, compute the answer and write it in the form $a + bi$.

27. $2i + (3 - i)$
28. $-3i + (2 - 5i)$
29. $2 + 3i + (3 - 2i)$
30. $(3 - 2i) - \left(2 + \frac{1}{2}i\right)$
31. $-3 - 5i - (2 - i)$
32. $\left(\frac{1}{2} - i\right) + \left(1 - \frac{2}{3}i\right)$
33. $-2i(3 + i)$
34. $3i(2 - i)$
35. $i\left(-\frac{1}{2} + i\right)$
36. $\frac{i}{2}\left(\frac{4 - i}{2}\right)$
37. $(2 - i)(2 + i)$
38. $(5 + i)(2 - 3i)$
39. $(-2 - 2i)(-4 - 3i)$
40. $(2 + 5i)(1 - 3i)$
41. $(3 - 2i)(2 - i)$
42. $(4 - 3i)(2 + 3i)$

In Exercises 43–48, multiply by the conjugate and simplify.

43. $2 - i$
44. $3 + i$
45. $3 + 4i$
46. $2 - 3i$
47. $-4 - 2i$
48. $5 + 2i$

In Exercises 49–57, perform the indicated operations and write the answer in the form $a + bi$.

49. $\frac{2 + 5i}{1 - 3i}$
50. $\frac{1 + 3i}{2 - 5i}$
51. $\frac{3 - 4i}{3 + 4i}$
52. $\frac{4 - 3i}{4 + 3i}$
53. $\frac{3 - 2i}{2 - i}$
54. $\frac{2 - 3i}{3 - i}$
55. $\frac{2 + 5i}{3i}$
56. $\frac{5 - 2i}{-3i}$
57. $\frac{4i}{2 + i}$

In Exercises 58–64, find the reciprocal and write the answer in the form $a + bi$.

58. $3 + 2i$
59. $4 + 3i$
60. $\frac{1}{2} - i$
61. $1 - \frac{1}{3}i$
62. $-7i$
63. $-5i$
64. $\frac{3 - i}{3 + 2i}$

In Exercises 65–68, evaluate the polynomial $x^2 - 2x + 5$ for the given complex value of x.

65. $1 + 2i$
66. $2 - i$
67. $1 - i$
68. $1 - 2i$
69. Prove that the commutative law of addition holds for the set of complex numbers.
70. Prove that the commutative law of multiplication holds for the set of complex numbers.
71. Prove that $0 + 0i$ is the additive identity and $1 + 0i$ is the multiplicative identity for the set of complex numbers.
72. Prove that $-a - bi$ is the additive inverse of the complex number $a + bi$.
73. Prove the distributive property for the set of complex numbers.
74. For what values of x is $\sqrt{x - 3}$ a real number?
75. For what values of y is $\sqrt{2y - 10}$ a real number?