If we invest P dollars at an annual interest rate of 6%, then we will earn $0.06P$ dollars interest per year; and we will have $P + 0.06P$ dollars at the end of the year. We call $P + 0.06P$ an **algebraic expression.** Note that an algebraic expression involves **variables** (in this case P), **constants** (such as 0.06), and **algebraic operations** (such as $+$, $-$, $\times$, $\div$). Virtually everything we do in algebra involves algebraic expressions.

An algebraic expression takes on a **value** when we assign a specific number to each variable in the expression. Thus, the expression

$$\frac{3m + 4n}{m + n}$$

is **evaluated** when $m = 3$ and $n = 2$ by substitution of these values for m and n:

$$\frac{3(3) + 4(2)}{3 + 2} = \frac{9 + 8}{5} = \frac{17}{5}$$

We often need to write algebraic expressions in which a variable multiplies itself repeatedly. We use the notation of exponents to indicate such repeated multiplication. Thus,

$$a^1 = a \qquad a^2 = a \cdot a \qquad a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factor}}$$

where n is a natural number and a is a real number. We call a the **base** and n the **exponent** and say that a^n is the nth **power** of a. When $n = 1$, we simply write a rather than a^1.

It is convenient to define a^0 for all real numbers $a \neq 0$ as $a^0 = 1$. We will provide motivation for this seemingly arbitrary definition in Section 1.7.

EXAMPLE 1 MULTIPLICATION WITH NATURAL NUMBER EXPONENTS

Write the following without using exponents.

a. $\left(\frac{1}{2}\right)^3$

b. $2x^3$

c. $(2x)^3$

d. $-3x^2y^3$

SOLUTION

a. $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

b. $2x^3 = 2 \cdot x \cdot x \cdot x$

c. $(2x)^3 = 2x \cdot 2x \cdot 2x = 8 \cdot x \cdot x \cdot x$

d. $-3x^2y^3 = -3 \cdot x \cdot x \cdot y \cdot y \cdot y$

WARNING

Note the difference between

$$(-3)^2 = (-3)(-3) = 9$$

and

$$-3^2 = -(3 \cdot 3) = -9$$

Calculator Alert

Your calculator evaluates exponents using a special key, which may be labeled $\boxed{x^y}$, $\boxed{y^x}$, or $\boxed{\wedge}$.

Example: $(1 \div 2)\boxed{x^y}3 = 0.125$

or $(1 \div 2)\boxed{y^x}3 = 0.125$

or $(1 \div 2)\boxed{\wedge}3 = 0.125$

We will use $\boxed{x^y}$ or $\boxed{\wedge}$ to indicate the exponentiation key in this text.

In addition to the exponentiation key, your calculator probably has a special key labeled $\boxed{x^2}$.

Examples: $(-3)\boxed{x^2} = 9$

$-3\boxed{x^2} = -9$

Later in this chapter we will need an important rule of exponents. Observe that if m and n are natural numbers and a is any real number, then

$$a^m \cdot a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Since there are a total of $m + n$ factors on the right side, we conclude that

$$a^m a^n = a^{m+n}$$

EXAMPLE 2 MULTIPLICATION WITH NATURAL NUMBER EXPONENTS

Multiply.

$$P = 4x^2 + 4x - 1$$

and

$$Q = 3x^3 - 2x^2 + 4$$

then the like terms are $0x^3$ and $3x^3$, $4x^2$ and $-2x^2$, $4x$ and $0x$, -1 and 4.

We define equality of polynomials in the following way:

> Two polynomials are equal if all like terms are equal.

EXAMPLE 6 EQUALITY OF POLYNOMIALS

Find A, B, C, and D if

$$Ax^3 + (A + B)x^2 + Cx + (C - D) = -2x^3 + x + 3$$

SOLUTION

Equating the coefficients of the terms, we have

$$A = -2 \qquad \begin{aligned} A + B &= 0 \\ B &= 2 \end{aligned} \qquad C = 1 \qquad \begin{aligned} C - D &= 3 \\ D &= -2 \end{aligned}$$

If P and Q are polynomials in x, the *sum* $P + Q$ is obtained by forming the sums of all pairs of like terms. The sum of ax^r in P and bx^r in Q is $(a + b)x^r$. Similarly, the *difference* $P - Q$ is obtained by forming the differences, $(a - b)x^r$, of like terms.

EXAMPLE 7 ADDITION AND SUBTRACTION OF POLYNOMIALS

a. Add $2x^3 + 2x^2 - 3$ and $x^3 - x^2 + x + 2$.

b. Subtract $2x^3 + x^2 - x + 1$ from $3x^3 - 2x^2 + 2x$.

SOLUTION

a. Adding the coefficients of like terms,

$$(2x^3 + 2x^2 - 3) + (x^3 - x^2 + x + 2) = 3x^3 + x^2 + x - 1$$

b. Subtracting the coefficients of like terms,

$$(3x^3 - 2x^2 + 2x) - (2x^3 + x^2 - x + 1) = x^3 - 3x^2 + 3x - 1$$

WARNING

$$(x + 5) - (x + 2) \neq (x + 5) - x + 2$$

The coefficient -1 must multiply each term in the parentheses. Thus,

$$-(x + 2) = -x - 2$$

Therefore,

$$(x + 5) - (x + 2) = x + 5 - x - 2 = 3$$

while

$$(x + 5) - x + 2 = x + 5 - x + 2 = 7$$

Multiplication of polynomials is based on the rule for exponents developed earlier in this section,

$$a^m a^n = a^{m+n}$$

and on the distributive laws

$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc$$

EXAMPLE 8 MULTIPLICATION OF POLYNOMIALS

Multiply $3x^3(2x^3 - 6x^2 + 5)$.

SOLUTION

$3x^3(2x^3 - 6x^2 + 5)$

$= (3x^3)(2x^3) + (3x^3)(-6x^2) + (3x^3)(5)$ Distributive law

$= (3)(2)x^{3+3} + (3)(-6)x^{3+2} + (3)(5)x^3$ $a^m a^n = a^{m+n}$

$= 6x^6 - 18x^5 + 15x^3$

EXAMPLE 9 MULTIPLICATION OF POLYNOMIALS

Multiply $(x + 2)(3x^2 - x + 5)$.

SOLUTION

$(x + 2)(3x^2 - x + 5)$

$= x(3x^2 - x + 5) + 2(3x^2 - x + 5)$ Distributive law

$= 3x^3 - x^2 + 5x + 6x^2 - 2x + 10$ Distributive law and $a^m a^n = a^{m+n}$

$= 3x^3 + 5x^2 + 3x + 10$ Adding like terms

✔ Progress Check

Multiply.

a. $(x^2 + 2)(x^2 - 3x + 1)$ b. $(x^2 - 2xy + y)(2x + y)$

Answers

a. $x^4 - 3x^3 + 3x^2 - 6x + 2$ b. $2x^3 - 3x^2y + 2xy - 2xy^2 + y^2$

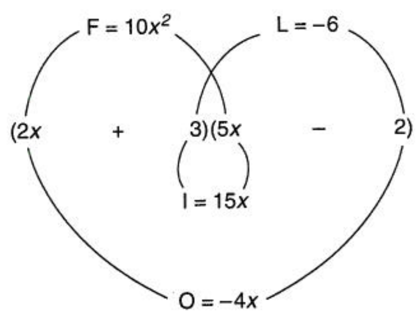
The multiplication in Example 9 can be carried out in "long form" as follows:

$$
\begin{array}{rl}
3x^2 - x + 5 & \\
x + 2 & \\
\hline
3x^3 - x^2 + 5x & = x(3x^2 - x + 5) \\
6x^2 - 2x + 10 & = 2(3x^2 - x + 5) \\
\hline
3x^3 + 5x^2 + 3x + 10 & = \text{sum of above lines}
\end{array}
$$

In Example 9, the product of polynomials of degrees 1 and 2 is seen to be a polynomial of degree 3. From the multiplication process, we can derive the following useful rule:

> The degree of the product of two nonzero polynomials is the sum of the degrees of the polynomials.

Products of the form $(2x + 3)(5x - 2)$ or $(2x + y)(3x - 2y)$ occur often, and we can handle them by the method sometimes referred to as FOIL: F = first, O = outer, I = inner, L = last.



$$
\begin{array}{rcl}
\text{F} & = & (2x)(5x) = 10x^2 \\
\text{O} & = & (2x)(-2) = -4x \\
\text{I} & = & (3)(5x) = 15x \\
\text{L} & = & (3)(-2) = -6 \\
\hline
\text{Sum} & = & 10x^2 - 4x + 15x - 6 \\
 & = & 10x^2 + 11x - 6
\end{array}
$$

A number of special products occur frequently, and it is worthwhile knowing them.

Special Products

$$(a + b)(a - b) = a^2 - b^2$$

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

EXAMPLE 10 MULTIPLICATION OF POLYNOMIALS

Multiply.

a. $(x + 2)^2$ b. $(x - 3)^2$ c. $(x + 4)(x - 4)$

SOLUTION

a. $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 4x + 4$

b. $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 6x + 9$

c. $(x + 4)(x - 4) = x^2 - 16$

✔ Progress Check

a. Multiply $(2x^2 - xy + y^2)(3x + y)$. b. Multiply $(2x - 3)(3x - 2)$.

Answers

a. $6x^3 - x^2y + 2xy^2 + y^3$ b. $6x^2 - 13x + 6$

Graphing Calculator Power User's Corner

(1) Assigning Values to Variables

Assign values to variables on your graphing calculator using the STORE command. There is usually an arrow key [→] or a key labeled [STO]. For example, to set $M = 3$, you press

[3] [→] [*M*] or [3] [STO] [*M*]

Check your owner's manual for details. The owner's manual may be available online. Look up your calculator by model and number.

(2) Evaluating Algebraic Expressions on a Graphing Calculator

Once specific values have been assigned to variables in your graphing calculator, you can use these variables to evaluate algebraic expressions. For example, evaluate

$$\frac{3m + 4n}{m + n}$$

when $m = 3$ and $n = 2$.

Step 1. Store 3 in memory location M and store 2 in memory location N.

Step 2. Enter and evaluate the expression

$$(3M + 4N) \div (M + N)$$

Note that the numerator expression and the denominator expression must both be enclosed in parentheses. On some calculators, you may need to enter $3 \times M$ and $4 \times N$ to multiply.

Step 3. Note that your answer is given in the decimal form 3.4. Use your calculator to verify that $\frac{17}{5} = 3.4$.

Exercise Set 1.3

In Exercises 1–6, evaluate the given expression when $r = 2$, $s = -3$, and $t = 4$.

1. $r + 2s + t$
2. rst
3. $\dfrac{rst}{r + s + t}$
4. $(r + s)t$
5. $\dfrac{r + s}{rt}$
6. $\dfrac{r + s + t}{t}$
7. Evaluate $\frac{2}{3}r + 5$ when $r = 12$.
8. Evaluate $\frac{9}{5}C + 32$ when $C = 37$.
9. If P dollars are invested at a simple interest rate of r percent per year for t years, the amount on hand at the end of t years is $P + Prt$. Suppose you invest \$2000 at 8% per year ($r = 0.08$). Find the amount you will have on hand after

   a. 1 year b. $\frac{1}{2}$ year c. 8 months.
10. The perimeter of a rectangle is given by the formula $P = 2(L + W)$, where L is the length and W is the width of the rectangle. Find the perimeter if

    a. $L = 2$ feet, $W = 3$ feet

    b. $L = \frac{1}{2}$ meter, $W = \frac{1}{4}$ meter
11. Evaluate $0.02r + 0.314st + 2.25t$ when $r = 2.5$, $s = 3.4$, and $t = 2.81$.
12. Evaluate $10.421x + 0.821y + 2.34xyz$ when $x = 3.21$, $y = 2.42$, and $z = 1.23$.

Evaluate the given expression in Exercises 13–18.

13. $|x| - |x| \cdot |y|$ when $x = -3, y = 4$
14. $|x + y| + |x - y|$ when $x = -3, y = 2$
15. $\dfrac{|a - 2b|}{2a}$ when $a = 1, b = 2$
16. $\dfrac{|x| + |y|}{|x| - |y|}$ when $x = -3, y = 4$
17. $\dfrac{-|a - 2b|}{|a + b|}$ when $a = -2, b = -1$
18. $\dfrac{|a - b| - 2|c - a|}{|a - b + c|}$ when $a = -2, b = 3, c = -5$

Carry out the indicated operations in Exercises 19–24.

19. $b^5 \cdot b^2$
20. $x^3 \cdot x^5$
21. $(4y^3)(-5y^6)$
22. $(-6x^4)(-4x^7)$
23. $\left(\frac{3}{2}x^3\right)(-2x)$
24. $\left(-\frac{5}{3}x^6\right)\left(-\frac{3}{10}x^3\right)$
25. Evaluate the given expressions and verify your answer using your calculator.

    a. 1^3 b. 10^8

    c. 2^5 d. 7^1
26. Evaluate the given expressions using your calculator.

    a. 9^{10} b. 0.8^6
27. Which of the following expressions are *not* polynomials?

    a. $-3x^2 + 2x + 5$ b. $-3x^2y$

    c. $-3x^{2/3} + 2xy + 5$ d. $-2x^{-4} + 2xy^3 + 5$

28. Which of the following expressions are *not* polynomials?

a. $4x^5 - x^{1/2} + 6$ b. $\frac{2}{5}x^3 + \frac{4}{3}x - 2$

c. $4x^5y$ d. $x^{4/3}y + 2x - 3$

In Exercises 29–32, indicate the leading coefficient and the degree of the given polynomial.

29. $2x^3 + 3x^2 - 5$

30. $-4x^5 - 8x^2 + x + 3$

31. $\frac{3}{5}x^4 + 2x^2 - x - 1$

32. $-1.5 + 7x^3 + 0.75x^7$

In Exercises 33–36, find the degree of the given polynomial.

33. $3x^2y - 4x^2 - 2y + 4$

34. $4xy^3 + xy^2 - y^2 + y$

35. $2xy^3 - y^3 + 3x^2 - 2$

36. $\frac{1}{2}x^3y^3 - 2$

37. Find the value of the polynomial $3x^2y^2 + 2xy - x + 2y + 7$ when $x = 2$ and $y = -1$.

38. Find the value of the polynomial $0.02x^2 + 0.3x - 0.5$ when $x = 0.3$.

39. Find the value of the polynomial $2.1x^3 + 3.3x^2 - 4.1x - 7.2$ when $x = 4.1$.

40. Write a polynomial giving the area of a circle of radius *r*.

41. Write a polynomial giving the area of a triangle of base *b* and height *h*.

42. A field consists of a rectangle and a square arranged as shown in Figure 4. What does each of the following polynomials represent?

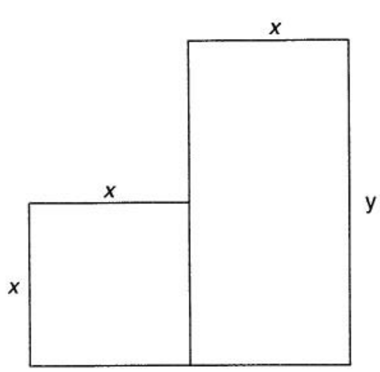


FIGURE 4 See Exercise 42.

a. $x^2 + xy$ b. $2x + 2y$

c. $4x$ d. $4x + 2y$

43. An investor buys *x* shares of G.E. stock at \$35.5 per share, *y* shares of Exxon stock at \$91 per share, and *z* shares of AT&T stock at \$38 per share. What does the polynomial $35.5x + 91y + 38z$ represent?

Perform the indicated operations in Exercises 44–62.

44. $(4x^2 + 3x + 2) + (3x^2 - 2x - 5)$

45. $(2x^2 + 3x + 8) - (5 - 2x + 2x^2)$

46. $4xy^2 + 2xy + 2x + 3 - (-2xy^2 + xy - y + 2)$

47. $(2s^2t^3 - st^2 + st - s + t) - (3s^2t^2 - 2s^2t - 4st^2 - t + 3)$

48. $3xy^2z - 4x^2yz + xy + 3 - (2xy^2z + x^2yz - yz + x - 2)$

49. $a^2bc + ab^2c + 2ab^3 - 3a^2bc - 4ab^3 + 3$

50. $(x + 1)(x^2 + 2x - 3)$

51. $(2 - x)(2x^3 + x - 2)$

52. $(2s - 3)(s^3 - s + 2)$

53. $(-3s + 2)(-2s^2 - s + 3)$

54. $(x^2 + 3)(2x^2 - x + 2)$

55. $(2y^2 + y)(-2y^3 + y - 3)$

56. $(x^2 + 2x - 1)(2x^2 - 3x + 2)$

57. $(a^2 - 4a + 3)(4a^3 + 2a + 5)$

58. $(2a^2 + ab + b^2)(3a - b^2 + 1)$

59. $(-3a + ab + b^2)(3b^2 + 2b + 2)$

60. $5(2x - 3)^2$

61. $2(3x - 2)(3 - x)$

62. $(x - 1)(x + 2)(x + 3)$

63. An investor buys *x* shares of IBM stock at \$98 per share at Thursday's opening of the stock market. Later in the day, the investor sells *y* shares of AT&T stock at \$38 per share and *z* shares of TRW stock at \$20 per share. Write a polynomial that expresses the

amount of money the buyer has invested at the end of the day.

64. An artist takes a rectangular piece of cardboard whose sides are x and y and cuts out a square of side $\frac{x}{2}$ to obtain a mat for a painting, as shown in Figure 5. Write a polynomial giving the area of the mat.

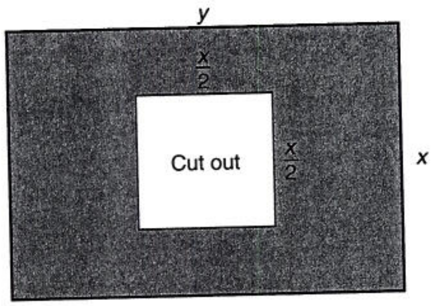


FIGURE 5 See Exercise 64.

In Exercises 65–78, perform the multiplication mentally.

65. $(x - 1)(x + 3)$
66. $(x + 2)(x + 3)$
67. $(2x + 1)(2x + 3)$
68. $(3x - 1)(x + 5)$
69. $(3x - 2)(x - 1)$
70. $(x + 4)(2x - 1)$
71. $(x + y)^2$
72. $(x - 4)^2$
73. $(3x - 1)^2$
74. $(x + 2)(x - 2)$
75. $(2x + 1)(2x - 1)$
76. $(3a + 2b)^2$
77. $(x^2 + y^2)^2$
78. $(x - y)^2$

79. Simplify the following.

a. $3^{10} + 3^{10} + 3^{10}$ b. $2^n + 2^n + 2^n + 2^n$

80. A student conjectured that the expression $N = m^2 - m + 41$ yields N, a prime number, for integer values of m. Prove or disprove this statement.

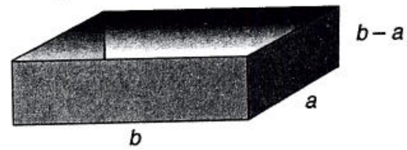
81. Perform the indicated operations.

a. $\left(\frac{2}{x} - 1\right)\left(\frac{2}{x} + 1\right)$

b. $\left(\frac{wx}{y} - z\right)^2$

c. $(x + y + z)(x + y - z)$

82. Find the surface area and volume of the open-top box below.



83. Eric can run a mile in 4.23 minutes, and Benjamin can run 4.23 miles in an hour. Who is the faster runner?

84. a. Let $P = \$1000$; that is, store 1000 in memory location P. Evaluate $P + 0.06P$ by entering the expression $P + 0.06P$ into your calculator.

b. Repeat part (a) for $P = \$28{,}525$.

85. Let $A = 8$ and $B = 32$; that is, store 8 in memory location A and 32 in memory location B. Evaluate the following expressions by entering them into your calculator as they appear below. (Use a multiplication sign if your calculator requires you to do so.)

a. $A(B + 17)$ b. $5B - A^2$

c. A^A d. $16^A - 3AB$

86. Find the value of the polynomial $20t - 0.7t^2$ when t is 28 and when t is 29. Try to find a value for t (other than 0) that gives the expression a value close to zero.

87. Consider the polynomial $vt - \frac{1}{2}at^2$.

a. Compare this expression to the expression given in Exercise 86. What values of v and a would make them identical?

b. Using your calculator, experiment with different values of v, a, and t. Try to put your data in an organized chart. In physics, this expression represents position of a body in free fall: v is the initial velocity, and a is the acceleration due to gravity.

1.4 Factoring

Now that we can find the product of two polynomials, let us consider the reverse problem: given a polynomial, can we find factors whose product yields the given polynomial? This process, known as **factoring**, is one of the basic tools of algebra. In this chapter, a polynomial with *integer* coefficients is to be factored as a product of polynomials of lower degree with *integer* coefficients; a polynomial with *rational* coefficients is to be factored as a product of polynomials of lower degree with *rational* coefficients. We will approach factoring by learning to recognize the situations in which factoring is possible.

Common Factors

Consider the polynomial

$$x^2 + x$$

Since the factor x is common to both terms, we can write

$$x^2 + x = x(x + 1)$$

EXAMPLE 1 FACTORING WITH COMMON FACTORS

Factor.

a. $15x^3 - 10x^2$ b. $4x^2y - 8xy^2 + 6xy$ c. $2x(x + y) - 5y(x + y)$

SOLUTION

a. 5 and x^2 are common to both terms. Therefore,

$$15x^3 - 10x^2 = 5x^2(3x - 2)$$

b. Here we see that 2, x, and y are common to all terms. Therefore,

$$4x^2y - 8xy^2 + 6xy = 2xy(2x - 4y + 3)$$

c. The expression $(x + y)$ is found in both terms. Factoring, we have

$$2x(x + y) - 5y(x + y) = (x + y)(2x - 5y)$$

✔ Progress Check

Factor.

a. $4x^2 - x$ b. $3x^4 - 9x^2$ c. $3m(2x - 3y) - n(2x - 3y)$

Answers

a. $x(4x - 1)$ b. $3x^2(x^2 - 3)$ c. $(2x - 3y)(3m - n)$

Factoring by Grouping

It is sometimes possible to discover common factors by first grouping terms. Consider the following examples:

EXAMPLE 2 FACTORING BY GROUPING

Factor.

a. $2ab + b + 2ac + c$

b. $2x - 4x^2y - 3y + 6xy^2$

SOLUTION

a. Group those terms containing b and those terms containing c.

$2ab + b + 2ac + c$	$= (2ab + b) + (2ac + c)$	Grouping
	$= b(2a + 1) + c(2a + 1)$	Common factors b, c
	$= (2a + 1)(b + c)$	Common factors $2a + 1$

Alternatively, suppose we group terms containing a.

$2ab + b + 2ac + c$	$= (2ab + 2ac) + (b + c)$	Grouping
	$= 2a(b + c) + (b + c)$	Common factors 2, a
	$= (b + c)(2a + 1)$	Common factor $(b + c)$

b. $2x - 4x^2y - 3y + 6xy^2$

$= (2x - 4x^2y) - (3y - 6xy^2)$	Grouping with sign change
$= 2x(1 - 2xy) - 3y(1 - 2xy)$	Common factors $2x$, $3y$
$= (1 - 2xy)(2x - 3y)$	Common factor $1 - 2xy$

✔ Progress Check

Factor.

a. $2m^3n + m^2 + 2mn^2 + n$

b. $2a^2 - 4ab^2 - ab + 2b^3$

Answers

a. $(2mn + 1)(m^2 + n)$

b. $(a - 2b^2)(2a - b)$

Factoring Second-Degree Polynomials

To factor a second-degree polynomial, such as

$$x^2 + 5x + 6$$

we first note that the term x^2 can have come only from $x \cdot x$, so we write two incomplete factors:

$$x^2 + 5x + 6 = (x \quad)(x \quad)$$

The constant term +6 can be the product of either two positive numbers or two negative numbers. Since the middle term $+5x$ is the sum of two other products, both signs must be positive. Thus,

$$x^2 + 5x + 6 = (x + \quad)(x + \quad)$$

Finally, the number 6 can be written as the product of two integers in only two ways: $1 \cdot 6$ and $2 \cdot 3$. The first pair gives a middle term of $7x$. The second pair gives the actual middle term, $5x$. So

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

EXAMPLE 3 **FACTORING SECOND-DEGREE POLYNOMIALS**

Factor.

a. $x^2 - 7x + 10$ b. $x^2 - 3x - 4$

SOLUTION

a. Since the constant term is positive and the middle term is negative, we must have two negative signs. Integer pairs whose product is 10 are 1 and 10, and 2 and 5. We find that

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

b. Since the constant term is negative, we must have opposite signs. Integer pairs whose product is 4 are 1 and 4, and 2 and 2. Since the coefficient of $-3x$ is negative, we assign the larger integer of a given pair to be negative. We find that

$$x^2 - 3x - 4 = (x + 1)(x - 4)$$

When the leading coefficient of a second-degree polynomial is an integer other than 1, the factoring process becomes more complex, as shown in the following example.

EXAMPLE 4 **FACTORING SECOND-DEGREE POLYNOMIALS**

Factor $2x^2 - x - 6$.

SOLUTION

The term $2x^2$ can result only from the factors $2x$ and x, so the factors must be of the form

$$2x^2 - x - 6 = (2x \quad)(x \quad)$$

The constant term, -6, must be the product of factors of opposite signs, so we may write

$$2x^2 - x - 6 = \begin{cases} (2x + \quad)(x - \quad) \\ \text{or} \\ (2x - \quad)(x + \quad) \end{cases}$$

The integer factors of 6 are

$$1 \cdot 6 \qquad 6 \cdot 1 \qquad 2 \cdot 3 \qquad 3 \cdot 2$$

By trying these we find that

$$2x^2 - x - 6 = (2x + 3)(x - 2)$$

✔ Progress Check

Factor:

a. $3x^2 - 16x + 21$ b. $2x^2 + 3x - 9$

Answers

a. $(3x - 7)(x - 3)$ b. $(2x - 3)(x + 3)$

WARNING

The polynomial $x^2 - 6x$ can be written as

$$x^2 - 6x = x(x - 6)$$

and is then a product of two polynomials of positive degree. Students often fail to consider x to be a "true" factor.

Special Factors

There is a special case of the second-degree polynomial that occurs frequently and factors easily. Given the polynomial $x^2 - 9$, we see that each term is a perfect square, and we can verify that

$$x^2 - 9 = (x + 3)(x - 3)$$

The general rule, which holds whenever we are dealing with a difference of two squares, may be stated as follows:

Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 5 SPECIAL FACTORS

Factor.

a. $4x^2 - 25$ b. $9r^2 - 16t^2$

SOLUTION

a. Since

$$4x^2 - 25 = (2x)^2 - (5)^2$$

we may use the formula for the difference of two squares with $a = 2x$ and $b = 5$. Thus,

$$4x^2 - 25 = (2x + 5)(2x - 5)$$

b. Since

$$9r^2 - 16t^2 = (3r)^2 - (4t)^2$$

we have $a = 3r$ and $b = 4t$, resulting in

$$9r^2 - 16t^2 = (3r + 4t)(3r - 4t)$$

✔ Progress Check

Factor.

a. $x^2 - 49$ b. $16x^2 - 9$ c. $25x^2 - y^2$

Answers

a. $(x + 7)(x - 7)$ b. $(4x + 3)(4x - 3)$ c. $(5x + y)(5x - y)$

The formulas for a sum of two cubes and a difference of two cubes can be verified by multiplying the factors on the right-hand sides of the following equations:

Sum and Difference of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These formulas provide a direct means of factoring the sum or difference of two cubes and are used in the same way as the formula for a difference of two squares.

EXAMPLE 6 SPECIAL FACTORS

Factor.

a. $x^3 + 1$ b. $27m^3 - 64n^3$ c. $\frac{1}{27}u^3 + 8v^3$

SOLUTION

a. With $a = x$ and $b = 1$, the formula for the sum of two cubes yields the following result:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

b. Since

$$27m^3 - 64n^3 = (3m)^3 - (4n)^3$$

we can use the formula for the difference of two cubes with $a = 3m$ and $b = 4n$:

$$27m^3 - 64n^3 = (3m - 4n)(9m^2 + 12mn + 16n^2)$$

c. Note that

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{1}{3}u\right)^3 + (2v)^3$$

and then use the formula for the sum of two cubes:

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{u}{3} + 2v\right)\left(\frac{u^2}{9} - \frac{2}{3}uv + 4v^2\right)$$

Combining Methods

We conclude with problems that combine the various methods of factoring that we have studied. As the factoring becomes more complicated, it may be helpful to consider the following strategy:

> Remove common factors before attempting any other factoring techniques.

EXAMPLE 7 COMMON FACTORS, GROUPING, AND SPECIAL FACTORS

Factor.

a. $2x^3 - 8x$ b. $3y(y + 3) + 2(y + 3)(y^2 - 1)$

SOLUTION

a. Observing the common factor $2x$, we find that

$$2x^3 - 8x = 2x(x^2 - 4)$$
$$= 2x(x + 2)(x - 2)$$

b. Observing the common factor $y + 3$, we see that

$$3y(y + 3) + 2(y + 3)(y^2 - 1) = (y + 3)[3y + 2(y^2 - 1)]$$
$$= (y + 3)(3y + 2y^2 - 2)$$
$$= (y + 3)(2y^2 + 3y - 2)$$
$$= (y + 3)(2y - 1)(y + 2)$$

Focus on "MAGICAL" Factoring for Second-Degree Polynomials

Factoring involves a certain amount of trial and error that can become frustrating, especially when the lead coefficient is not 1. You might want to try a scheme that "magically" reduces the number of candidates. We demonstrate the method for the polynomial

$$4x^2 + 11x + 6 \tag{1}$$

Using the lead coefficient of 4, write the pair of incomplete factors

$$(4x \qquad)(4x \qquad) \tag{2}$$

Next, multiply the coefficient of x^2 and the constant term in Equation (1) to produce $4 \cdot 6 = 24$. Now find two integers whose product is 24 and whose sum is 11, the coefficient of the middle term of (1). Since 8 and 3 work, we write

$$(4x + 8)(4x + 3) \tag{3}$$

Finally, within each parenthesis in Equation (3) discard any common numerical factor. (Discarding a factor may only be performed in this "magical" type of factoring.) Thus $(4x + 8)$ reduces to $(x + 2)$ and we write

$$(x + 2)(4x + 3) \tag{4}$$

which is the factorization of $4x^2 + 11x + 6$.

Will the method always work? Yes—if you first remove all common factors in the original polynomial. That is, you must first write

$$6x^2 + 15x + 6 = 3(2x^2 + 5x + 2)$$

and apply the method to the polynomial $2x^2 + 5x + 2$.

(For a proof that the method works, see M. A. Autrie and J. D. Austin, "A Novel Way to Factor Quadratic Polynomials," *The Mathematics Teacher* 72, no. 2 [1979].) We use the polynomial $2x^2 - x - 6$ of Example 4 to demonstrate the method when some of the coefficients are negative.

Try the method on these second-degree polynomials:

$3x^2 + 10x - 8$
$6x^2 - 13x + 6$
$4x^2 - 15x - 4$
$10x^2 + 11x - 6$

Factoring $ax^2 + bx + c$	Example: $2x^2 - x - 6$
Step 1. Use the lead coefficient a to write the incomplete factors $(ax \quad)(ax \quad)$	*Step 1.* The lead coefficient is 2, so we write $(2x \quad)(2x \quad)$
Step 2. Multiply a and c, the coefficients of x^2, and the constant term.	*Step 2.* $a \cdot c = (2)(-6) = -12$
Step 3. Find integers whose product is $a \cdot c$ and whose sum equals b. Write these integers in the incomplete factors of *Step 1*.	*Step 3.* Two integers whose product is -12 and whose sum is -1 are 3 and -4. We then write $(2x + 3)(2x - 4)$
Step 4. Discard any common factor *within each parenthesis* in *Step 3*. The result is the desired factorization.	*Step 4.* Reducing $(2x - 4)$ to $(x - 2)$ by discarding the common factor 2, we have $2x^2 - x - 6 = (2x + 3)(x - 2)$

✓ Progress Check

Factor.

a. $x^3 + 5x^2 - 6x$ b. $2x^3 - 2x^2y - 4xy^2$

c. $-3x(x + 1) + (x + 1)(2x^2 + 1)$

Answers

a. $x(x + 6)(x - 1)$ b. $2x(x + y)(x - 2y)$ c. $(x + 1)(2x - 1)(x - 1)$

Irreducible Polynomials

Are there polynomials that cannot be written as a product of polynomials of lower degree with integer coefficients? The answer is yes. Examples are the polynomials $x^2 + 1$ and $x^2 + x + 1$. A polynomial is said to be **prime**, or **irreducible**, if it cannot be written as a product of two polynomials, each of positive degree. Thus, $x^2 + 1$ is irreducible over the integers.

Exercise Set 1.4

Factor completely.

1. $5x - 15$
2. $\frac{1}{4}x + \frac{3}{4}y$
3. $-2x - 8y$
4. $3x - 6y + 15$
5. $5bc + 25b$
6. $2x^4 + x^2$
7. $-3y^2 - 4y^5$
8. $3abc + 12bc$
9. $3x^2 + 6x^2y - 9x^2z$
10. $9a^3b^3 + 12a^2b - 15ab^2$
11. $x^2 + 4x + 3$
12. $x^2 + 2x - 8$
13. $y^2 - 8y + 15$
14. $y^2 + 7y - 8$
15. $a^2 - 7ab + 12b^2$
16. $x^2 - 49$
17. $y^2 - \frac{1}{9}$
18. $a^2 - 7a + 10$

19. $9 - x^2$

20. $4b^2 - a^2$

21. $x^2 - 5x - 14$

22. $x^2y^2 - 9$

23. $\frac{1}{16} - y^2$

24. $4a^2 - b^2$

25. $x^2 - 6x + 9$

26. $a^2b^2 - \frac{1}{9}$

27. $x^2 - 12x + 20$

28. $x^2 - 8x - 20$

29. $x^2 + 11x + 24$

30. $y^2 - \frac{9}{16}$

31. $2x^2 - 3x - 2$

32. $2x^2 + 7x + 6$

33. $3a^2 - 11a + 6$

34. $4x^2 - 9x + 2$

35. $6x^2 + 13x + 6$

36. $4y^2 - 9$

37. $8m^2 - 6m - 9$

38. $9x^2 + 24x + 16$

39. $10x^2 - 13x - 3$

40. $9y^2 - 16x^2$

41. $6a^2 - 5ab - 6b^2$

42. $4x^2 + 20x + 25$

43. $10r^2s^2 + 9rst + 2t^2$

44. $x^{12} - 1$

45. $16 - 9x^2y^2$

46. $6 + 5x - 4x^2$

47. $8n^2 - 18n - 5$

48. $15 + 4x - 4x^2$

49. $2x^2 - 2x - 12$

50. $3y^2 + 6y - 45$

51. $30x^2 - 35x + 10$

52. $x^4y^4 - x^2y^2$

53. $18x^2m + 33xm + 9m$

54. $25m^2n^3 - 5m^2n$

55. $12x^2 - 22x^3 - 20x^4$

56. $10r^2 - 5rs - 15s^2$

57. $x^4 - y^4$

58. $a^4 - 16$

59. $b^4 + 2b^2 - 8$

60. $4b^4 + 20b^2 + 25$

61. $x^3 + 27y^3$

62. $8x^3 + 125y^3$

63. $27x^3 - y^3$

64. $64x^3 - 27y^3$

65. $a^3 + 8$

66. $8r^3 - 27$

67. $\frac{1}{8}m^3 - 8n^3$

68. $8a^3 - \frac{1}{64}b^3$

69. $(x + y)^3 - 8$

70. $27 + (x + y)^3$

71. $8x^6 - 125y^6$

72. $a^6 + 27b^6$

73. $4(x + 1)(y + 2) - 8(y + 2)$

74. $2(x + 1)(x - 1) + 5(x - 1)$

75. $3(x + 2)^2(x - 1) - 4(x + 2)^2(2x + 7)$

76. $4(2x - 1)^2(x + 2)^3(x + 1) - 3(2x - 1)^5(x + 2)^2(x + 3)$

77. Show that the difference of the squares of two positive, consecutive odd integers must be divisible by 8.

78. A perfect square is a natural number of the form n^2. For example, 9 is a perfect square since $9 = 3^2$. Show that the sum of the squares of two odd numbers cannot be a perfect square.

79. Find a natural number n, if possible, such that $1 + n(n + 1)(n + 2)(n + 3)$ is a perfect square.

80. Prove or disprove that $1 + n(n + 1)(n + 2)(n + 3)$ is a perfect square. (*Hint:* Consider $[1 + n(n + 3)]^2$.)

81. Factor completely.

a. $(x + h)^3 - x^3$

b. $2^n + 2^{n+1} + 2^{n+2}$

c. $16 - 81x^{12}$

d. $z^2 - x^2 + 2xy - y^2$

82. Factor completely.

a. $\left[\frac{n(n + 1)}{2}\right]^2 + (n + 1)^3$

b. $\frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2$

c. $\frac{1}{b}(a + bx)^2 - \frac{a}{b}(a + bx)$

83. Factor the following expressions that arise in different branches of science.

a. biology (blood flow): $C[(R + 1)^2 - r^2]$

b. physics (nuclear): $pa^2 + (1 - p)b^2 - [pa + (1 - p)b]^2$

c. mechanics (bending beams): $X^2 - 3LX + 2L^2$

d. electricity (resistance): $(R_1 + R_2)^2 - 2r(R_1 + R_2)$

e. physics (motion): $-16t^2 + 64t + 336$

84. a. Factor this expression, used to find the answers given in the chapter opening.

$$20t - 0.7t^2$$