1.1 The Real Number System

Sets

We will need to use the notation and terminology of sets from time to time. A set is simply a collection of objects or numbers that are called the **elements** or **members** of the set. The elements of a set are written within braces so that the notation

$$A = \{4, 5, 6\}$$

tells us that the set A consists of the numbers 4, 5, and 6. The set

$$B = \{\text{Exxon, Ford, Sony}\}$$

consists of the names of these three corporations. We also write $4 \in A$, which we read as "4 is a member of the set A." Similarly, Ford $\in B$ is read as "Ford is a member of the set B," and Chrysler $\notin B$ is read as "Chrysler is not a member of the set B."

If every element of a set A is also a member of a set B, then A is a subset of B. For example, the set of all robins is a subset of the set of all birds.

EXAMPLE 1 SET NOTATION AND PROPERTIES

The set C consists of the names of all coins whose denominations are less than 50 cents. We may write C in set notation as follows:

$$C = \{\text{penny, nickel, dime, quarter}\}$$

We see that dime $\in C$, but half dollar $\notin C$. Further, the set $H = \{\text{nickel, dime}\}$ is a subset of C.

✔ Progress Check

The set V consists of the vowels in this particular sentence.

a. Write V in set notation.
b. Is the letter k a member of V?
c. Is the letter u a member of V?
d. List the subsets of V having four elements.

Answers

a. $V = \{a, e, i, o, u\}$ b. No c. Yes
d. $\{a, e, i, o\}, \{e, i, o, u\}, \{a, i, o, u\}, \{a, e, o, u\}, \{a, e, i, u\}$

The Set of Real Numbers

Since much of our work in algebra deals with the real numbers, we begin with a review of the composition of these numbers.

The numbers 1, 2, 3, . . . , used for counting, form the set of **natural numbers**. If we had only these numbers to use to show the profit earned by a company, we would have no way to indicate that the company had no profit or had a loss. To indicate no profit we introduce 0, and for losses we need to introduce negative numbers. The numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

form the set of **integers**. Thus, every natural number is an integer, and the set of natural numbers is seen to be a subset of the set of integers.

When we try to divide two apples equally among four people, we find no number in the set of integers that expresses how many apples each person should get. We need to introduce the set of **rational numbers**, which are numbers that can be written as a ratio of two integers.

$$\frac{p}{q} \text{ with } q \text{ not equal to zero}$$

Examples of rational numbers are

$$0 \quad \frac{2}{3} \quad -4 \quad \frac{7}{5} \quad \frac{-3}{4}$$

By writing an integer n in the form $\frac{n}{1}$, we see that every integer is a rational number. The decimal number 1.3 is also a rational number since $1.3 = \frac{13}{10}$.

We have now seen three fundamental sets of numbers: the set of natural numbers, the set of integers, and the set of rational numbers. Each successive set includes the previous set or sets, and each is more complicated than the one before. However, the set of rational numbers is still inadequate for sophisticated uses of mathematics, since there exist numbers that are not rational, that is, numbers that cannot be written as the ratio of two integers. These are called **irrational numbers**. It can be shown that the number a that satisfies $a \cdot a = 2$ is such a number. The number π, which is the ratio of the circumference of a circle to its diameter, is also such a number.

The decimal form of a rational number always forms a repeating pattern, such as

$$\frac{1}{2} = 0.5000\ldots$$

$$\frac{13}{10} = 1.3000\ldots$$

$$\frac{1}{3} = 0.333\ldots$$

$$\frac{-2}{11} = -0.\underbrace{18}\underbrace{18}\underbrace{18}\ldots$$

(*Note*: The three dots, known as ellipses, following the numbers in each of the examples above means that the pattern continues in the same manner forever.)

The decimal form of an irrational number never forms a repeating pattern. The rational and irrational numbers together form the set of **real numbers**. (See Figure 1.)

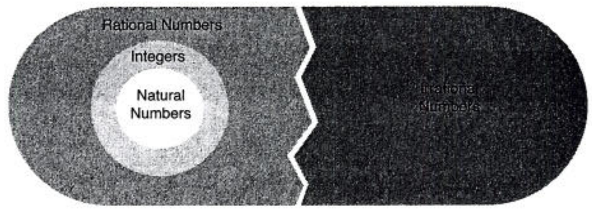


FIGURE 1 The Set of Real Numbers

Calculator Alert

(1) Rational Numbers

A calculator display shows only a finite number of digits, which is often an approximation of the exact answer. Use your calculator to convert the rational numbers $\frac{1}{2}$, $\frac{13}{10}$, $\frac{1}{3}$, and $\frac{-2}{11}$ to decimal form, and note how these representations differ from those shown above.

$$1 \div 2 = 0.5$$
$$13 \div 10 = 1.3$$
$$1 \div 3 = 0.3333333333$$
$$-2 \div 11 = -0.1818181818$$

Which representations are exact and which are approximate?

(2) Irrational Numbers

Most calculators provide a rational decimal *approximation* to irrational numbers. For example, ten-digit approximations to $\sqrt{2}$ and π are

$$\sqrt{2} = 1.414213562$$
$$\pi = 3.141592654$$

The System of Real Numbers

The system of real numbers consists of the set of real numbers together with the operations of addition and multiplication; in addition, this system satisfies the properties listed in Table 1, where a, b, and c denote real numbers.

EXAMPLE 2 PROPERTIES OF REAL NUMBERS

Specify the property in Table 1 illustrated by each of the following statements.

a. $2 + 3 = 3 + 2$

b. $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

c. $2 \cdot \frac{1}{2} = 1$

d. $2(3 + 5) = (2 \cdot 3) + (2 \cdot 5)$

SOLUTION

a. commutative under addition
b. associative under multiplication
c. multiplicative inverse
d. distributive law

Equality

When we say that two numbers are **equal**, we mean that they represent the same value. Thus, when we write

$$a = b$$

Read "a equals b", we mean that a and b represent the same number. For example, 2 + 5 and 4 + 3 are different ways of writing the number 7, so we can write

$$2 + 5 = 4 + 3$$

Equality satisfies four basic properties shown in Table 2, where a, b, and c are any real numbers.

TABLE 1 Properties of Real Numbers

Example	Algebraic Expression	Property
3 + 4 is a real number.	$a + b$ is a real number.	**Closure under addition** The sum of two real numbers is a real number.
$2 \cdot 5$ is a real number.	$a \cdot b$ is a real number.	**Closure under multiplication** The product of two real numbers is a real number.
$4 + 8 = 8 + 4$	$a + b = b + a$	**Commutative under addition** We may add real numbers in any order.
$3(5) = 5(3)$	$a(b) = b(a)$	**Commutative under multiplication** We may multiply real numbers in any order.
$(2 + 5) + 3 = 2 + (5 + 3)$	$(a + b) + c = a + (b + c)$	**Associative under addition** We may group the addition of real numbers in any order.
$(2 \cdot 5)3 = 2(5 \cdot 3)$	$(ab)c = a(bc)$	**Associative under multiplication** We may group the multiplication of real numbers in any order.
$4 + 0 = 4$	$a + 0 = a$	**Additive identity** The sum of the unique real number 0 and any real number leaves that number unchanged.
$3(1) = 3$	$a(1) = a$	**Multiplicative identity** The product of the unique real number 1 and any real number leaves that number unchanged.
$5 + (-5) = 0$	$a + (-a) = 0$	**Additive inverse** The number $-a$ is called the **negative, opposite,** or **additive inverse** of a. If $-a$ is added to a, the result is the additive identity 0.

continues

TABLE 1 Properties of Real Numbers (cont.)

Example	Algebraic Expression	Property
$7\left(\frac{1}{7}\right) = 1$	If $a \neq 0$, $a\left(\frac{1}{a}\right) = 1$	**Multiplicative inverse** The number $\frac{1}{a}$ is called the **reciprocal,** or **multiplicative inverse,** of a. If $\frac{1}{a}$ is multiplied by a, the result is the multiplicative identity 1.
$2(5 + 3) = (2 \cdot 5) + (2 \cdot 3)$ $(4 + 7)2 = (4 \cdot 2) + (7 \cdot 2)$	$a(b + c) = ab + ac$ $(a + b)c = ac + bc$	**Distributive laws** If one number multiplies the sum of two numbers, we may add the two numbers first and then perform the multiplication; or we may multiply each pair and then add the two products.

EXAMPLE 3 PROPERTIES OF EQUALITY

Specify the property in Table 2 illustrated by each of the following statements.

a. If $5a - 2 = b$, then $b = 5a - 2$.

b. If $a = b$ and $b = 5$, then $a = 5$.

c. If $a = b$, then $3a + 6 = 3b + 6$.

SOLUTION

a. symmetric property b. transitive property c. substitution property

Additional Properties

Using the properties of real numbers, the properties of equalities, and rules of logic, we can derive many other properties of the real numbers, as shown in Table 3, where a, b, and c are any real numbers.

TABLE 2 Properties of Equality

Example	Algebraic Expression	Property
$3 = 3$	$a = a$	Reflexive property
If $\frac{6}{3} = 2$ then $2 = \frac{6}{3}$.	If $a = b$ then $b = a$.	Symmetric property
If $\frac{6}{3} = 2$ and $2 = \frac{8}{4}$, then $\frac{6}{3} = \frac{8}{4}$	If $a = b$ and $b = c$, then $a = c$.	Transitive property
If $\frac{6}{3} = 2$, then we may replace $\frac{6}{3}$ by 2 or we may replace 2 by $\frac{6}{3}$.	If $a = b$, then we may replace a by b or we may replace b by a.	Substitution property

TABLE 3 Additional Properties of Real Numbers

Example	Algebraic Expression	Property
If $\frac{6}{3} = 2$ then $\frac{6}{3} + 4 = 2 + 4$ $\frac{6}{3}(5) = 2(5)$	If $a = b$, then $a + c = b + c$ $ac = bc$	The same number may be added to both sides of an equation. Both sides of an equation may be multiplied by the same number.
If $\frac{6}{3} + 4 = 2 + 4$ then $\frac{6}{3} = 2$.	If $a + c = b + c$ then $a = b$.	Cancellation law of addition
If $\frac{6}{3}(5) = 2(5)$ then $\frac{6}{3} = 2$.	If $ac = bc$ with $c \neq 0$ then $a = b$.	Cancellation law of multiplication
$2(0) = 0(2) = 0$ $2(3) = 0$ is impossible.	$a(0) = 0(a) = 0$ If $ab = 0$ then $a = 0$ or $b = 0$.	The product of two real numbers can be zero only if one of them is zero. The real numbers a and b are said to be **factors** of the product ab.
$-(-3) = 3$ $(-2)(3) = (2)(-3) = -6$ $(-1)(3) = -3$ $(-2)(-3) = 6$ $(-2) + (-3) = -(2 + 3) = -5$	$-(-a) = a$ $(-a)(b) = (a)(-b) = -(ab)$ $(-1)(a) = -a$ $(-a)(-b) = ab$ $(-a) + (-b) = -(a + b)$	Rules of signs

We next introduce the operations of subtraction and division. If a and b are real numbers, the *difference* between a and b, denoted by $a - b$, is defined by

$$a - b = a + (-b)$$

and the operation is called *subtraction*. Thus,

$$6 - 2 = 6 + (-2) = 4 \qquad 2 - 2 = 0 \qquad 0 - 8 = -8$$

We can show that the distributive laws hold for subtraction, that is,

$$a(b - c) = ab - ac$$

$$(a - b)c = ac - bc$$

Calculator Alert

In addition to a key for subtraction, most calculators have a key to represent negative numbers. This key may be labeled [+/−], [(−)], or [CHS]. We write −6, for example, to represent the keystrokes necessary to enter a negative number into your calculator. You must select the keystrokes that are appropriate for your calculator.

If a and b are real numbers and $b \neq 0$, then the *quotient* of a and b, denoted $\frac{a}{b}$ or a/b, is defined by

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

and the operation is called *division*. We also write $\frac{a}{b}$ as $a \div b$ and speak of the *fraction a* over b. The numbers a and b are called the *numerator* and *denominator* of the fraction $\frac{a}{b}$, respectively. Observe that we have not defined division by zero, since 0 has no reciprocal.

In Table 4, a, b, c, and d are real numbers with $b \neq 0$, $c \neq 0$, and $d \neq 0$.

✔ Progress Check

Perform the indicated operations.

a. $\frac{3}{5} + \frac{1}{4}$ b. $\frac{5}{2} \cdot \frac{4}{15}$ c. $\frac{2}{3} + \frac{3}{7}$

Answers

a. $\frac{17}{20}$ b. $\frac{2}{3}$ c. $\frac{23}{21}$

TABLE 4 Additional Properties of Real Numbers

Example	Algebraic Expression	Property
$\frac{6}{10} = \frac{2 \cdot 3}{2 \cdot 5} = \frac{3}{5}$	$\frac{ac}{bc} = \frac{a}{b}$	Rules of fractions
$\frac{2}{3} \cdot \frac{5}{7} = \frac{10}{21}$	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	
$\frac{4}{6} = \frac{2}{3}$ since $4 \cdot 3 = 6 \cdot 2$	$\frac{a}{b} = \frac{c}{d}$ if $ad = bc$	
$\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$	$\frac{a}{d} + \frac{c}{d} = \frac{a+c}{d}$	
$\frac{\frac{2}{9}}{\frac{5}{9}} = \frac{\frac{2}{9}}{\frac{5}{9}} \cdot \frac{(9)}{(9)} = \frac{2}{5}$	$\frac{\frac{a}{d}}{\frac{b}{d}} = \frac{\frac{a}{d}}{\frac{b}{d}} \cdot \frac{(d)}{(d)} = \frac{a}{b}$	
$\frac{2}{3} + \frac{5}{4} = \frac{2}{3} \cdot \frac{(4)}{(4)} + \frac{5}{4} \cdot \frac{(3)}{(3)}$ $= \frac{8}{12} + \frac{15}{12} = \frac{23}{12}$	$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{(d)}{(d)} + \frac{c}{d} \cdot \frac{(b)}{(b)}$ $= \frac{ad + cb}{bd}$	
$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3}}{\frac{5}{7}} \cdot \frac{(3 \cdot 7)}{(3 \cdot 7)} = \frac{2(7)}{5(3)} = \frac{14}{15}$	$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{(bd)}{(bd)} = \frac{ad}{cb}$	

Exercise Set 1.1

In Exercises 1–8, write each set by listing its elements within braces.

1. The set of natural numbers from 3 to 7, inclusive
2. The set of integers between -4 and 2
3. The set of integers between -10 and -8
4. The set of natural numbers from -9 to 3, inclusive
5. The subset of the set $S = \{-3, -2, -1, 0, 1, 2\}$ consisting of the positive integers in S
6. The subset of the set $S = \{-\frac{2}{3}, -1.1, 3.7, 4.8\}$ consisting of the negative rational numbers in S
7. The subset of all $x \in S$, $S = \{1, 3, 6, 7, 10\}$, such that x is an odd integer
8. The subset of all $x \in S$, $S = \{2, 5, 8, 9, 10\}$ such that x is an even integer

In Exercises 9–22, determine whether the given statement is true (T) or false (F).

9. -14 is a natural number.
10. $-\frac{4}{5}$ is a rational number.
11. $\frac{\pi}{3}$ is a rational number.
12. $\frac{1.75}{18.6}$ is an irrational number.
13. -1207 is an integer.
14. 0.75 is an irrational number.
15. $\frac{4}{5}$ is a real number.
16. 3 is a rational number.
17. 2π is a real number.
18. The sum of two rational numbers is always a rational number.
19. The sum of two irrational numbers is always an irrational number.
20. The product of two rational numbers is always a rational number.
21. The product of two irrational numbers is always an irrational number.
22. The difference of two irrational numbers is always an irrational number.

In Exercises 23–36, the letters represent real numbers. Identify the property or properties of real numbers that justify each statement.

23. $a + x = x + a$
24. $(xy)z = x(yz)$
25. $xyz + xy = xy(z + 1)$
26. $x + y$ is a real number
27. $(a + b) + 3 = a + (b + 3)$
28. $5 + (x + y) = (x + y) + 5$
29. cx is a real number.
30. $(a + 5) + b = (a + b) + 5$
31. $uv = vu$
32. $x + 0 = x$
33. $a(bc) = c(ab)$
34. $xy - xy = 0$
35. $5 \cdot \frac{1}{5} = 1$
36. $xy \cdot 1 = xy$

In Exercises 37–40, find a counterexample; that is, find real values for which the statement is false.

37. $a - b = b - a$
38. $\frac{a}{b} = \frac{b}{a}$
39. $a(b + c) = ab + c$
40. $(a + b)(c + d) = ac + bd$

In Exercises 41–44, indicate the property or properties of equality that justify the statement.

41. If $3x = 5$, then $5 = 3x$.
42. If $x + y = 7$ and $y = 5$, then $x + 5 = 7$.
43. If $2y = z$ and $z = x + 2$, then $2y = x + 2$.
44. If $x + 2y + 3z = r + s$ and $r = x + 1$, then $x + 2y + 3z = x + 1 + s$.

In Exercises 45–49, a, b, and c are real numbers. Use the properties of real numbers and the properties of equality to prove each statement.

45. If $-a = -b$, then $ac = bc$.

46. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

47. If $a - c = b - c$, then $a = b$.

48. $a(b - c) = ab - ac$

49. Prove that the real number 0 does not have a reciprocal. (*Hint:* Assume $b = \frac{1}{0}$ is the reciprocal of 0.) Supply a reason for each of the following steps.

$$1 = 0 \cdot \frac{1}{0}$$
$$= 0 \cdot b$$
$$= 0$$

Since this conclusion is impossible, the original assumption must be false.

50. Give three examples for each of the following:
   a. a real number that is not a rational number
   b. a rational number that is not an integer
   c. an integer that is not a natural number

51. Give three examples for each of the following:
   a. two rational numbers that are not integers whose sum is an integer
   b. two irrational numbers whose sum is a rational number

52. Find a subset of the reals that is closed with respect to addition and multiplication but not with respect to subtraction and division.

53. Perform the indicated operations. Verify your answers using your calculator.
   a. $(-8) + 13$
   b. $(-8) + (-13)$
   c. $8 - (-13)$
   d. $(-5)(3) - (-12)$
   e. $\left(\frac{8}{9} + 3\right) + \left(\frac{-5}{9}\right)$
   f. $\frac{-5}{\frac{3}{2}}$
   g. $\frac{\frac{5}{8}}{\frac{1}{2}}$
   h. $\frac{\frac{-2}{3}}{\frac{-4}{3}}$
   i. $\left(\frac{3}{4}\right)\left(\frac{21}{37}\right) + \left(\frac{3}{4}\right)\left(\frac{16}{37}\right)$
   j. $\dfrac{\frac{1}{3} - \left(\frac{-1}{4}\right)}{\frac{7}{8} - \frac{3}{16}}$
   k. $\dfrac{\left(\frac{3}{5}\right)\left(\frac{1}{7}\right)}{\frac{1}{2} + \frac{1}{3}}$
   l. $\frac{2}{5}\left(\frac{3}{2} \cdot \frac{4}{7}\right)$

54. What is the meaning attached to each of the following?
   a. $\frac{6}{0}$
   b. $\frac{0}{6}$
   c. $\frac{6}{6}$
   d. $\frac{0}{\frac{1}{2}}$
   e. $\frac{0}{0}$

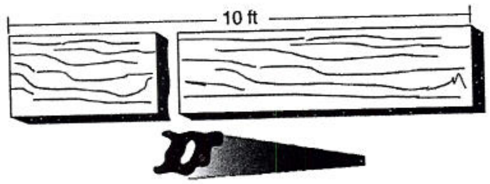
55. Use your calculator to convert the following fractions to (repeating) decimals. Look for a pattern that repeats.
   a. $\frac{1}{4}$ b. $-\frac{3}{5}$ c. $\frac{10}{13}$ d. $\frac{2}{7}$
   e. Does your calculator round off the final digit of an approximation, or does your calculator "drop off" the extra digits? To answer this question, evaluate $2 \div 3$ to see if your calculator displays 0.6666666666 or 0.6666666667.

56. A proportion is a statement of equality between two ratios. Solve the following proportions for x.
   a. $\frac{7}{8} = \frac{x}{12}$
   b. $\frac{7}{x} = \frac{11}{3}$

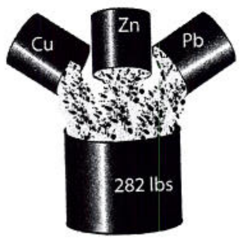
57. On a map of Pennsylvania, 1 inch represents 10 miles. Find the distance represented by 3.5 inches.

58. A car travels 135 miles on 6 gallons of gasoline. How far can it travel on 10 gallons of gasoline?

59. A board 10 feet long is cut into two pieces, the lengths of which are in the ratio of 2:3. Find the lengths of the pieces.



60. An alloy is $\frac{3}{8}$ copper, $\frac{5}{12}$ zinc, and the balance lead. How much lead is there in 282 pounds of alloy?



61. Which is the better value: 1 pound 3 ounces of beans for 85 cents or 13 ounces for 56 cents?

62. A piece of property is valued at \$28,500. What is the real estate tax at \$75.30 per \$1000.00 evaluation?

63. A woman's take-home pay is \$210.00 after deducting 18% withholding tax. What is her pay before the deduction?

64. List the set of possible ways of getting a total of 7 when tossing two standard dice.

65. A college student sent a postcard home with the following message:

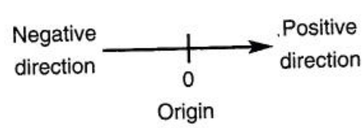
$$\begin{array}{r} \text{SEND} \\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

If each letter represents a different digit, and the calculation represents a sum, how much money did the student request?

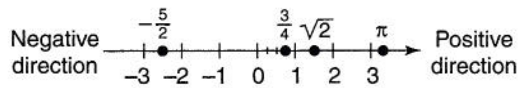
66. Eric starts at a certain time driving his car from New York to Philadelphia going 50 mph. Sixty minutes later, Steve leaves in his car en route from Philadelphia to New York going 40 mph. When the two cars meet, which one is nearer to New York?

1.2 The Real Number Line

There is a simple and very useful geometric interpretation of the real number system. Draw a horizontal line. Pick a point on this line, label it with the number 0, and call it the **origin**. Designate the side to the right of the origin as the *positive direction* and the side to the left as the *negative direction*.



Next, select a unit for measuring distance. With each positive integer n, we associate the point that is n units to the right of the origin. With each negative number $-n$, we associate the point that is n units to the left of the origin. Rational numbers, such as $\frac{3}{4}$ and $-\frac{5}{2}$, are associated with the corresponding points by dividing the intervals between integers into equal subintervals. Irrational numbers, such as $\sqrt{2}$ and π, can be written in decimal form. The corresponding points can be found by approximating these decimal forms to any desired degree of accuracy. Thus, the set of real numbers is identified with all possible points on this line. There is a real number for every point on the line; there is a point on the line for every real number. The line is called the **real number line**, and the number associated with a point is called its *coordinate*. We can now show some points on this line.



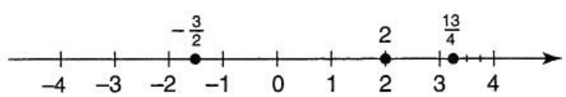
The numbers to the right of zero are called *positive*; the numbers to the left of zero are called *negative*. The positive numbers and zero together are called the **nonnegative numbers.**

We will frequently use the real number line to help picture the results of algebraic computations. For this purpose, we are only concerned with relative locations on the line. For example, it is adequate to show π slightly to the right of 3 since π is approximately 3.14.

EXAMPLE 1 REAL NUMBER LINE

Draw a real number line and plot the following points: $-\frac{3}{2}$, 2, $\frac{13}{4}$.

SOLUTION



Inequalities

If a and b are real numbers, we can compare their positions on the real number line by using the relations *less than, greater than, less than or equal to*, and *greater than or equal to*, as shown in Table 5.

TABLE 5 Inequalities

Symbol	Meaning
$<$	Less than
$>$	Greater than
$\leq$	Less than or equal to
$\geq$	Greater than or equal to

Table 6 describes both algebraic and geometric interpretations of the inequality symbols, where a and b are real numbers.

Expressions involving inequality symbols, such as $a < b$ and $a \geq b$, are called **inequalities.** We often combine these expressions so that $a \leq b < c$ means both $a \leq b$ and $b < c$. (*Note*: $a < c$ is also true.) For example, $-5 \leq x < 2$ is equivalent to $-5 \leq x$ and $x < 2$. Equivalently, x is between -5 and 2, including -5 and excluding 2.

✔ Progress Check

Verify that the following inequalities are true by using either the "Equivalent Statement" or the "Geometric Statement" of Table 6.

a. $-1 < 3$

b. $2 \le 2$

c. $-2.7 < -1.2$

d. $-4 < -2 < 0$

e. $-\frac{7}{2} < \frac{7}{2} < 7$

TABLE 6 Inequalities

Algebraic Expression	Meaning	Equivalent Statement	Geometric Statement
$a > 0$	a is greater than 0.	a is positive.	a lies to the right of the origin.
$a < 0$	a is less than 0.	a is negative.	a lies to the left of the origin.
$a > b$	a is greater than b.	$a - b$ is positive.	a lies to the right of b.
$a < b$	a is less than b.	$a - b$ is negative.	a lies to the left of b.
$a \ge b$	a is greater than or equal to b.	$a - b$ is positive or zero.	a lies to the right of b or coincides with b.
$a \le b$	a is less than or equal to b.	$a - b$ is negative or zero.	a lies to the left of b or coincides with b.

TABLE 7 Properties of Inequalities

Example	Algebraic Expression	Property
Either $2 < 3$, $2 > 3$, or $2 = 3$.	Either $a < b$, $a > b$, or $a = b$.	Trichotomy property
Since $2 < 3$ and $3 < 5$, then $2 < 5$.	If $a < b$ and $b < c$ then $a < c$.	Transitive property
Since $2 < 5$, then $2 + 4 < 5 + 4$ or $6 < 9$.	If $a < b$ then $a + c < b + c$.	The sense of an inequality is preserved if any constant is added to both sides.
Since $2 < 3$ and $4 > 0$, then $2(4) < 3(4)$ or $8 < 12$.	If $a < b$ and $c > 0$, then $ac < bc$.	The sense of an inequality is preserved if it is multiplied by a positive constant.
Since $2 < 3$ and $-4 < 0$, then $2(-4) > 3(-4)$ or $-8 > -12$.	If $a < b$ and $c < 0$, then $ac > bc$.	The sense of an inequality is reversed if it is multiplied by a negative constant.

The real numbers satisfy the properties of inequalities shown in Table 7, where a, b, and c are real numbers.

EXAMPLE 2 PROPERTIES OF INEQUALITIES

a. Since $-2 < 4$ and $4 < 5$, then $-2 < 5$.

b. Since $-2 < 5$, $-2 + 3 < 5 + 3$, or $1 < 8$.

c. Since $3 < 4$, $3 + (-5) < 4 + (-5)$, or $-2 < -1$.

d. Since $2 < 5$, $2(3) < 5(3)$, or $6 < 15$.

e. Since $-3 < 2$, $(-3)(-2) > 2(-2)$, or $6 > -4$.

Absolute Value

Suppose we are interested in the *distances* between the origin and the points labeled 4 and −4 on the real number line. Each of these points is four units from the origin; that is, the *distance is independent of the direction* and is nonnegative. (See Figure 2.) Furthermore, the distance between 4 and −4 is 8 units.

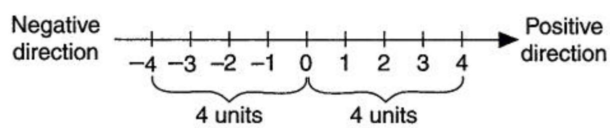


FIGURE 2 Distance on the Real Number Line

When we are interested in the magnitude of a number a, and do not care about the direction or sign, we use the concept of **absolute value**, which we write as $|a|$. The formal definition of absolute value is stated as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Since distance is independent of direction and is always nonnegative, we can view $|a|$ as the distance from the origin to either point a or point $-a$ on the real number line.

EXAMPLE 3 ABSOLUTE VALUE AND DISTANCE

a. $|4| = 4$ $\quad |-4| = 4$ $\quad |0| = 0$

b. The distance on the real number line between the point labeled 3.4 and the origin is $|3.4| = 3.4$. Similarly, the distance between point -2.3 and the origin is $|-2.3| = 2.3$.

In working with the notation of absolute value, it is important to perform the operations within the bars first. Here are some examples.

EXAMPLE 4 ABSOLUTE VALUE

a. $|5 - 2| = |3| = 3$

b. $|2 - 5| = |-3| = 3$

c. $|3 - 5| - |8 - 6| = |-2| - |2| = 2 - 2 = 0$

d. $\frac{|4-7|}{-6} = \frac{|-3|}{-6} = \frac{3}{-6} = -\frac{1}{2}$

Graphing Calculator Alert

Your calculator may have an absolute value key, usually labeled ABS. If you have a graphing calculator, it is important to use parentheses when you use this key.

Examples:

a. ABS(5 − 2)

b. ABS(2 − 5)

c. ABS(3 − 5) − ABS(8 − 6)

d. ABS(4 − 7) ÷ (−6)

Table 8 describes the properties of absolute value where a and b are real numbers.

We began by showing a use for absolute value in denoting distance from the origin without regard to direction. We conclude by demonstrating the use of absolute value to denote the distance between *any* two points a and b on the real number line. In Figure 3, the distance between the points labeled 2 and 5 is 3 units and can be obtained by evaluating either $|5 - 2|$ or $|2 - 5|$. Similarly, the distance between the points labeled −1 and 4 is given by either $|4 - (-1)| = 5$ or $|-1 - 4| = 5$.

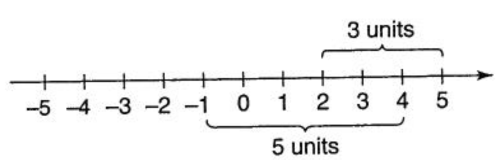


FIGURE 3 Distance on the Real Number Line

TABLE 8 Basic Properties of Absolute Value

Example	Algebraic Expression	Property
$\lvert -2 \rvert \geq 0$	$\lvert a \rvert \geq 0$	Absolute value is always nonnegative.
$\lvert 3 \rvert = \lvert -3 \rvert = 3$	$\lvert a \rvert = \lvert -a \rvert$	The absolute value of a number and its negative are the same.
$\lvert 2-5 \rvert = \lvert -3 \rvert = 3$ $\lvert 5-2 \rvert = \lvert 3 \rvert = 3$	$\lvert a-b \rvert = \lvert b-a \rvert$	The absolute value of the difference of two numbers is always the same, irrespective of the order of subtraction.
$\lvert (-2)(3) \rvert = \lvert -2 \rvert \lvert 3 \rvert = 6$	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.

Using the notation $\overline{AB}$ to denote the distance between the points A and B, we provide the following definition:

Distance on the Real Number Line

The **distance** $\overline{AB}$ between points A and B on the real number line, whose coordinates are a and b, respectively, is given by

$$\overline{AB} = \lvert b - a \rvert$$

The third property of absolute value from Table 8 tells us that $\overline{AB} = \lvert b - a \rvert = \lvert a - b \rvert$. Viewed another way, this property states that the distance between any two points on the real number line is independent of the direction.

EXAMPLE 5 DISTANCE ON THE REAL NUMBER LINE

Let points A, B, and C have coordinates -4, -1, and 3, respectively, on the real number line. Find the following distances.

a. $\overline{AB}$ b. $\overline{CB}$ c. $\overline{OB}$, where O is the origin

SOLUTION

Using the definition, we have

a. $\overline{AB} = \lvert -1 - (-4) \rvert = \lvert -1 + 4 \rvert = \lvert 3 \rvert = 3$

b. $\overline{CB} = \lvert -1 - 3 \rvert = \lvert -4 \rvert = 4$

c. $\overline{OB} = \lvert -1 - 0 \rvert = \lvert -1 \rvert = 1$

✔ Progress Check

The points P, Q, and R on the real number line have coordinates −6, 4, and 6, respectively. Find the following distances.

a. $\overline{PR}$ b. $\overline{QP}$ c. $\overline{PQ}$

Answers

a. 12 b. 10 c. 10

Exercise Set 1.2

1. Draw a real number line and plot the following points.

   a. 4 b. -2

   c. $\frac{5}{2}$ d. -3.5

   e. 0

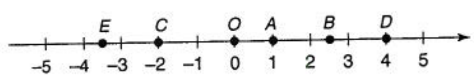
2. Draw a real number line and plot the following points.

   a. -5 b. 4

   c. -3.5 d. $\frac{7}{2}$

   e. -4

3. Give the real numbers associated with the points A, B, C, D, O, and E on the real number line below.



4. Represent the following by real numbers.

   a. a profit of \$10

   b. a loss of \$20

   c. a temperature of 20°F above zero

   d. a temperature of 5°F below zero

In Exercises 5–10, indicate which of the two given numbers appears first, viewed from left to right, on the real number line.

5. 4, 6 6. $\frac{1}{2}$, 0

7. $-2, \frac{3}{4}$ 8. 0, -4

9. $-5, -\frac{2}{3}$ 10. 4, -5

In Exercises 11–14, indicate the set of numbers on a real number line.

11. The natural numbers less than 8

12. The natural numbers greater than 4 and less than 10

13. The integers that are greater than 2 and less than 7

14. The integers that are greater than -5 and less than or equal to 1

In Exercises 15–24, express the statement as an inequality.

15. 10 is greater than 9.99.

16. -6 is less than -2.

17. a is nonnegative.

18. b is negative.

19. x is positive.

20. a is strictly between 3 and 7.

21. a is strictly between $\frac{1}{2}$ and $\frac{1}{4}$.

22. b is less than or equal to -4.

23. b is greater than or equal to 5.

24. x is negative.

In Exercises 25–30, give a property of inequalities that justifies the statement.

25. Since $-3 < 1$, then $-1 < 3$.

26. Since $-5 < -1$ and $-1 < 4$, then $-5 < 4$.

27. Since $14 > 9$, then $-14 < -9$.

28. Since $5 > 3$, then $5 \neq 3$.

29. Since $-1 < 6$, then $-3 < 18$.

30. Since $6 > -1$, then 7 is a positive number.

In Exercises 31–44, find the value of the expression. Verify your answer using your calculator.

31. $|2|$

32. $\left|-\frac{2}{3}\right|$

33. $|1.5|$

34. $|-0.8|$

35. $-|2|$

36. $-\left|-\frac{2}{5}\right|$

37. $|2 - 3|$

38. $|2 - 2|$

39. $|2 - (-2)|$

40. $|2| + |-3|$

41. $\dfrac{|14 - 8|}{|-3|}$

42. $\dfrac{|2 - 12|}{|1 - 6|}$

43. $\dfrac{|3| - |2|}{|3| + |2|}$

44. $\dfrac{|3 - 2|}{|3 + 2|}$

In Exercises 45–50, the coordinates of points A and B are given. Find $\overline{AB}$.

45. 2, 5

46. -3, 6

47. -3, -1

48. -4, $\frac{11}{2}$

49. $-\frac{4}{5}$, $\frac{4}{5}$

50. 2, 2

51. For what values of x and y is $|x + y| = |x| + |y|$?

52. For what values of x and y is $|x + y| < |x| + |y|$?

53. Find the set of integers whose distance from 3 is less than or equal to 5.

54. List the set of integers x such that

a. $-2 < x < 3$

b. $0 < x < 5$

c. $-1 < 2x < 10$

55. For the inequality $-1 < 5$, state the resulting inequality when the following operations are performed on both sides.

a. add 2

b. subtract 5

c. multiply by 2

d. multiply by -5

e. divide by -1

f. divide by 2

g. square

56. A computer sales representative receives \$400 monthly plus a 10% commission on sales. How much must she sell in a month for her income to be at least \$600 for that month?

In Exercises 57–62, use the coordinates given in Exercises 45–50 to find the midpoint of the interval.

63. For what values of x does each of the following hold?

a. $|3 - x| = 3 - x$

b. $|5x - 2| = -(5x - 2)$

64. Evaluate $\dfrac{|x - 3|}{x - 3}$ for $x = -2, -1, 0, 1, 2$. Make a conjecture about the value of this expression for all values of x.

1.3 Algebraic Expressions and Polynomials

A **variable** is a symbol to which we can assign values. For example, in Section 1.1 we defined a rational number as one that can be written as $\frac{p}{q}$, where p and q are integers and q is not zero. The symbols p and q are variables since we can assign values to them. A variable can be restricted to a particular number system (for example, p and q must be integers) or to a subset of a number system.