The Difference of Two Squares 6.4

In Chapter 5 we listed the following three special products:

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$
$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Since factoring is the reverse of multiplication, we can also consider the three special products as three special factorings:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

Any trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ can be factored by the methods of Section 4.6. The last line is the factoring to obtain the difference of two squares. The difference of two squares always factors in this way. Again, these are patterns you must be able to recognize on sight.

EXAMPLE 1 Factor $16x^2 - 25$.

SOLUTION We can see that the first term is a perfect square, and the last term is also. This fact becomes even more obvious if we rewrite the problem as:

$$16x^2 - 25 = (4x)^2 - (5)^2$$

The first term is the square of the quantity $4x$, and the last term is the square of 5. The completed problem looks like this:

$$\begin{aligned} 16x^2 - 25 &= (4x)^2 - (5)^2 \\ &= (4x + 5)(4x - 5) \end{aligned}$$

To check our results, we multiply:

$$\begin{aligned} (4x + 5)(4x - 5) &= 16x^2 + 20x - 20x - 25 \\ &= 16x^2 - 25 \end{aligned}$$

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EXAMPLE 2 Factor $36a^2 - 1$.

SOLUTION We rewrite the two terms to show they are perfect squares and then factor. Remember, 1 is its own square, $1^2 = 1$.

$$\begin{aligned} 36a^2 - 1 &= (6a)^2 - (1)^2 \\ &= (6a + 1)(6a - 1) \end{aligned}$$

To check our results, we multiply:

$$\begin{aligned} (6a + 1)(6a - 1) &= 36a^2 + 6a - 6a - 1 \\ &= 36a^2 - 1 \end{aligned}$$

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EXAMPLE 3 Factor $x^4 - y^4$.

SOLUTION x^4 is the perfect square $(x^2)^2$, and y^4 is $(y^2)^2$:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2$$
$$= (x^2 - y^2)(x^2 + y^2)$$

The factor $(x^2 - y^2)$ is itself the difference of two squares and therefore can be factored again. The factor $(x^2 + y^2)$ is the *sum* of two squares and cannot be factored again. The complete problem is this:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2$$
$$= (x^2 - y^2)(x^2 + y^2)$$
$$= (x + y)(x - y)(x^2 + y^2)$$

Note: If you think the sum of two squares $x^2 + y^2$ factors, you should try it. Write down the factors you think it has, and then multiply them using the foil method. You won't get $x^2 + y^2$.

EXAMPLE 4 Factor $25x^2 - 60x + 36$.

SOLUTION Although this trinomial can be factored by the method we used in Section 6.3, we notice that the first and last terms are the perfect squares $(5x)^2$ and $(6)^2$. Before going through the method for factoring trinomials by listing all possible factors, we can check to see if $25x^2 - 60x + 36$ factors to $(5x - 6)^2$. We need only multiply to check:

$$(5x - 6)^2 = (5x - 6)(5x - 6)$$
$$= 25x^2 - 30x - 30x + 36$$
$$= 25x^2 - 60x + 36$$

The trinomial $25x^2 - 60x + 36$ factors to $(5x - 6)(5x - 6) = (5x - 6)^2$.

EXAMPLE 5 Factor $5x^2 + 30x + 45$.

SOLUTION We begin by factoring out the greatest common factor, which is 5. Then we notice that the trinomial that remains is a perfect square trinomial:

$$5x^2 + 30x + 45 = 5(x^2 + 6x + 9)$$
$$= 5(x + 3)^2$$

Note: As we have indicated before, perfect square trinomials like the ones in Examples 4 and 5 can be factored by the methods developed in previous sections. Recognizing that they factor to binomial squares simply saves time in factoring.

EXAMPLE 6 Factor $(x - 3)^2 - 25$.

SOLUTION This example has the form $a^2 - b^2$, where a is $x - 3$ and b is 5. We factor it according to the formula for the difference of two squares:

$$(x - 3)^2 - 25 = (x - 3)^2 - 5^2 \quad \text{Write 25 as } 5^2$$
$$= [(x - 3) - 5][(x - 3) + 5] \quad \text{Factor}$$
$$= (x - 8)(x + 2) \quad \text{Simplify}$$

Notice in this example we could have expanded $(x - 3)^2$, subtracted 25, and then factored to obtain the same result:

$$(x - 3)^2 - 25 = x^2 - 6x + 9 - 25 \quad \text{Expand } (x \cdot 3)^2$$
$$= x^2 - 6x - 16 \quad \text{Simplify}$$
$$= (x - 8)(x + 2) \quad \text{Factor}$$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

A. Describe how you factor the difference of two squares.
B. What is a perfect square trinomial?
C. How do you know when you've factored completely?
D. Describe how you would factor $25x^2 - 60x + 36$.

SPOTLIGHT ON SUCCESS *Student Instructor Stefanie*

Never confuse a single defeat with a final defeat.
—F. Scott Fitzgerald

The idea that has worked best for my success in college, and more specifically in my math courses, is to stay positive and be resilient. I have learned that a 'bad' grade doesn't make me a failure; if anything it makes me strive to do better. That is why I never let a bad grade on a test or even in a class get in the way of my overall success.

By sticking with this positive attitude, I have been able to achieve my goals. My grades have never represented how well I know the material. This is because I have struggled with test anxiety and it has consistently lowered my test scores in a number of courses. However, I have not let it defeat me. When I applied to graduate school, I did not meet the grade requirements for my top two schools, but that did not stop me from applying.

One school asked that I convince them that my knowledge of mathematics was more than my grades indicated. If I had let my grades stand in the way of my goals, I wouldn't have been accepted to both of my top two schools, and will be attending one of them in the Fall, on my way to becoming a mathematics teacher.

Problem Set 6.4

Factor the following.

1. $x^2 - 9$ **2.** $x^2 - 25$ **3.** $a^2 - 36$ **4.** $a^2 - 64$

5. $x^2 - 49$ **6.** $x^2 - 121$ **7.** $4a^2 - 16$ **8.** $4a^2 + 16$

9. $9x^2 + 25$ **10.** $16x^2 - 36$ **11.** $25x^2 - 169$ **12.** $x^2 - y^2$

13. $9a^2 - 16b^2$ **14.** $49a^2 - 25b^2$ **15.** $9 - m^2$ **16.** $16 - m^2$

17. $25 - 4x^2$ **18.** $36 - 49y^2$ **19.** $2x^2 - 18$ **20.** $3x^2 - 27$

21. $32a^2 - 128$ **22.** $3a^3 - 48a$ **23.** $8x^2y - 18y$ **24.** $50a^2b - 72b$

25. $a^4 - b^4$ **26.** $a^4 - 16$ **27.** $16m^4 - 81$ **28.** $81 - m^4$

29. $3x^3y - 75xy^3$ **30.** $2xy^3 - 8x^3y$

Factor the following.

31. $x^2 - 2x + 1$ **32.** $x^2 - 6x + 9$

33. $x^2 + 2x + 1$ **34.** $x^2 + 6x + 9$

35. $a^2 - 10a + 25$ **36.** $a^2 + 10a + 25$

37. $y^2 + 4y + 4$ **38.** $y^2 - 8y + 16$

39. $x^2 - 4x + 4$ **40.** $x^2 + 8x + 16$

41. $m^2 - 12m + 36$ **42.** $m^2 + 12m + 36$

43. $4a^2 + 12a + 9$ **44.** $9a^2 - 12a + 4$

45. $49x^2 - 14x + 1$ **46.** $64x^2 - 16x + 1$

47. $9y^2 - 30y + 25$ **48.** $25y^2 + 30y + 9$

49. $x^2 + 10xy + 25y^2$ **50.** $25x^2 + 10xy + y^2$

51. $9a^2 + 6ab + b^2$ **52.** $9a^2 - 6ab + b^2$

Factor the following by first factoring out the greatest common factor.

53. $3a^2 + 18a + 27$ **54.** $4a^2 - 16a + 16$

55. $2x^2 + 20xy + 50y^2$ **56.** $3x^2 + 30xy + 75y^2$

57. $5x^3 + 30x^2y + 45xy^2$ **58.** $12x^2y - 36xy^2 + 27y^3$

Factor by grouping the first three terms together.

59. $x^2 + 6x + 9 - y^2$ **60.** $x^2 + 10x + 25 - y^2$

61. $x^2 + 2xy + y^2 - 9$ **62.** $a^2 + 2ab + b^2 - 25$

63. Find a value for b so that the polynomial $x^2 + bx + 49$ factors to $(x + 7)^2$.

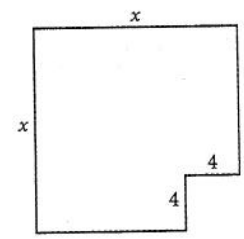
64. Find a value of b so that the polynomial $x^2 + bx + 81$ factors to $(x + 9)^2$.

65. Find the value of c for which the polynomial $x^2 + 10x + c$ factors to $(x + 5)^2$.

66. Find the value of a for which the polynomial $ax^2 + 12x + 9$ factors to $(2x + 3)^2$.

Applying the Concepts

67. Area

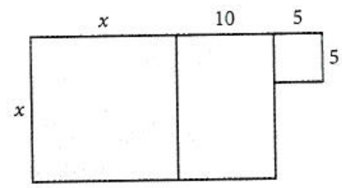


a. What is the area of the following figure?

b. Factor the answer from part **a**.

c. Find a way to cut the figure into two pieces and put them back together to show that the factorization in part **b**. is correct.

68. Area



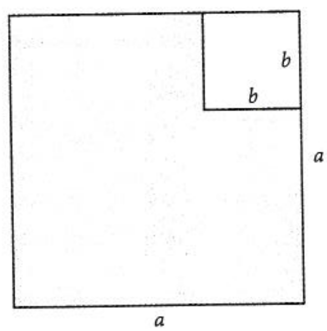
a. What is the area of the following figure?

b. Factor the expression from part **a**.

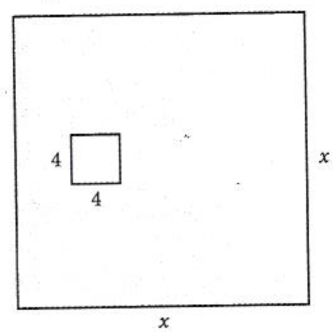
c. Cut and rearrange the figure to show that the factorization is correct.

Find the area for the shaded regions; then write your result in factored form.

69.



70.



Getting Ready for the Next Section

Multiply each of the following:

71. a. 1^3 **b.** 2^3 **c.** 3^3 **d.** 4^3 **e.** 5^3

72. a. $(-1)^3$ **b.** $(-2)^3$ **c.** $(-3)^3$ **d.** $(-4)^3$ **e.** $(-5)^3$

73. a. $x(x^2 - x + 1)$ **b.** $1(x^2 - x + 1)$ **c.** $(x + 1)(x^2 - x + 1)$

74. a. $x(x^2 + x + 1)$ **b.** $-1(x^2 + x + 1)$ **c.** $(x - 1)(x^2 + x + 1)$

75. a. $x(x^2 - 2x + 4)$ **b.** $2(x^2 - 2x + 4)$ **c.** $(x + 2)(x^2 - 2x + 4)$

76. a. $x(x^2 + 2x + 4)$ **b.** $-2(x^2 + 2x + 4)$ **c.** $(x - 2)(x^2 + 2x + 4)$

77. a. $x(x^2 - 3x + 9)$ **b.** $3(x^2 - 3x + 9)$ **c.** $(x + 3)(x^2 - 3x + 9)$

78. a. $x(x^2 + 3x + 9)$ **b.** $-3(x^2 + 3x + 9)$ **c.** $(x - 3)(x^2 + 3x + 9)$