29. Use the formula for finding the sum

$$1 + 2 + 3 + \cdots + n$$

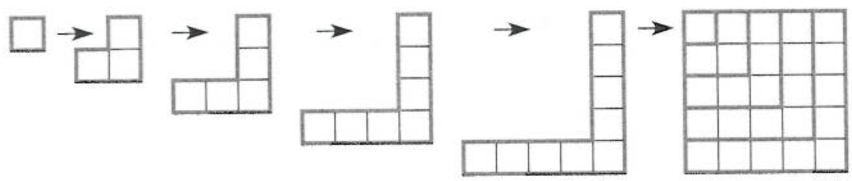
to discover a formula for finding the sum

$$2 + 4 + 6 + \cdots + 2n.$$ $S = n(n + 1)$

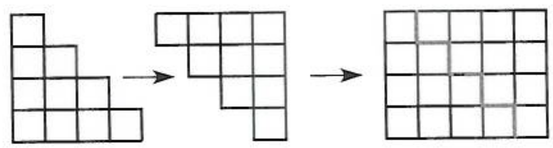
30. State in your own words the following formula discussed in this section: Answers will vary.

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

31. Explain how the following diagram geometrically illustrates the formula $1 + 3 + 5 + 7 + 9 = 5^2$. Answers will vary.



32. Explain how the following diagram geometrically illustrates the formula $1 + 2 + 3 + 4 = \frac{4 \times 5}{2}$. Answers will vary.



33. Use patterns to complete the table below.

Figurate Number	1st	2nd	3rd	4th	5th	6th	7th	8th
Triangular	1	3	6	10	15	21	28	36
Square	1	4	9	16	25	36	49	64
Pentagonal	1	5	12	22	35	51	70	92
Hexagonal	1	6	15	28	45	66	91	120
Heptagonal	1	7	18	34	55	81	112	148
Octagonal	1	8	21	40	65	96	133	176

34. The first five triangular, square, and pentagonal numbers may be obtained using sums of terms of sequences, as shown below.

Triangular	Square	Pentagonal
$1 = 1$	$1 = 1$	$1 = 1$
$3 = 1 + 2$	$4 = 1 + 3$	$5 = 1 + 4$
$6 = 1 + 2 + 3$	$9 = 1 + 3 + 5$	$12 = 1 + 4 + 7$
$10 = 1 + 2 + 3 + 4$	$16 = 1 + 3 + 5 + 7$	$22 = 1 + 4 + 7 + 10$
$15 = 1 + 2 + 3 + 4 + 5$	$25 = 1 + 3 + 5 + 7 + 9$	$35 = 1 + 4 + 7 + 10 + 13$

Notice the successive differences of the added terms on the right sides of the equations. The next type of figurate number is the **hexagonal** number. (A hexagon has six sides.) Use the patterns above to predict the first five hexagonal numbers. 1, 6, 15, 28, 45

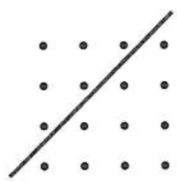
35. Eight times any triangular number, plus 1, is a square number. Show that this is true for the first four triangular numbers.

36. Divide the first triangular number by 3 and record the remainder. Divide the second triangular number by 3 and record the remainder. Repeat this procedure several more times. Do you notice a pattern?

37. Repeat Exercise 36, but instead use square numbers and divide by 4. What pattern is determined?

38. Exercises 36 and 37 are specific cases of the following: In general, when the numbers in the sequence of n-agonal numbers are divided by n, the sequence of remainders obtained is a repeating sequence. Verify this for $n = 5$ and $n = 6$.

39. Every square number can be written as the sum of two triangular numbers. For example, $16 = 6 + 10$. This can be represented geometrically by dividing a square array of dots with a line as shown.



The triangular arrangement above the line represents 6, the one below the line represents 10, and the whole arrangement represents 16. Show how the square numbers 25 and 36 may likewise be geometrically represented as the sum of two triangular numbers.

40. A fraction is in *lowest terms* if the greatest common factor of its numerator and its denominator is 1. For example, $\frac{3}{8}$ is in lowest terms, but $\frac{4}{12}$ is not.

(a) For $n = 2$ to $n = 8$, form the fractions

$$\frac{n\text{th square number}}{(n+1)\text{th square number}}$$

(b) Repeat part (a), but use triangular numbers instead.

(c) Use inductive reasoning to make a conjecture based on your results from parts (a) and (b), observing whether the fractions are in lowest terms.

41. Complete the following table.

n	2	3	4	5	6	7	8
A Square of n							
B (Square of n) + n							
C One-half of Row B entry							
D (Row A entry) – n							
E One-half of Row D entry							

Use your results to answer the following, using inductive reasoning.

(a) What kind of figurate number is obtained when you find the average of n^2 and n? (See Row C.) a triangular number

(b) If you square n and then subtract n from the result, and then divide by 2, what kind of figurate number is obtained? (See Row E.) a triangular number

42. Find the least integer N greater than 1 such that two different figurate numbers equal N. What are they?

In addition to the formulas for T_n, S_n, and P_n, the following formulas are true for ***hexagonal*** *numbers (H),* ***heptagonal*** *numbers (Hp), and* ***octagonal*** *numbers (O):*

$$H_n = \frac{n(4n-2)}{2}, \quad Hp_n = \frac{n(5n-3)}{2}, \quad O_n = \frac{n(6n-4)}{2}.$$

Use these formulas to find each of the following.

43. the sixteenth square number 256

44. the eleventh triangular number 66

45. the ninth pentagonal number 117

46. the seventh hexagonal number 91

47. the tenth heptagonal number 235

48. the twelfth octagonal number 408

49. Observe the formulas given for H_n, Hp_n, and O_n, and use patterns and inductive reasoning to predict the formula for N_n, the nth **nonagonal** number. (A nonagon has nine sides.) Then use the fact that the sixth nonagonal number is 111 to further confirm your conjecture. $N_n = \frac{n(7n-5)}{2}$

50. Use the result of Exercise 49 to find the tenth nonagonal number. 325

Use inductive reasoning to answer each question.

51. If you add two consecutive triangular numbers, what kind of figurate number do you get? a square number

52. If you add the squares of two consecutive triangular numbers, what kind of figurate number do you get? a triangular number

53. Square a triangular number. Square the next triangular number. Subtract the smaller result from the larger. What kind of number do you get? a perfect cube

54. Choose a value of n greater than or equal to 2. Find T_{n-1}, multiply it by 3, and add n. What kind of figurate number do you get? a pentagonal number

1.3 Strategies for Problem Solving

A General Problem-Solving Method • Using a Table or Chart • Working Backward • Using Trial and Error • Guessing and Checking • Considering a Similar Simpler Problem • Drawing a Sketch • Using Common Sense

George Polya, author of the classic *How to Solve it,* died at the age of 97 on September 7, 1985. A native of Budapest, Hungary, he was once asked why there were so many good mathematicians to come out of Hungary at the turn of the century. He theorized that it was because mathematics is the cheapest science. It does not require any expensive equipment, only pencil and paper. He authored or coauthored more than 250 papers in many languages, wrote a number of books, and was a brilliant lecturer and teacher. Yet, interestingly enough, he never learned to drive a car.

A General Problem-Solving Method In the first two sections of this chapter we stressed the importance of pattern recognition and the use of inductive reasoning in solving problems. There are other useful approaches. These ideas are used throughout the text.

Probably the most famous study of problem-solving techniques was developed by George Polya (1888–1985), among whose many publications was the modern classic *How to Solve It*. In this book, Polya proposed a four-step method for problem solving.

Polya's Four-Step Method for Problem Solving

Step 1 **Understand the problem.** You cannot solve a problem if you do not understand what you are asked to find. The problem must be read and analyzed carefully. You may need to read it several times. After you have done so, ask yourself, "What must I find?"

Step 2 **Devise a plan.** There are many ways to attack a problem. Decide what plan is appropriate for the particular problem you are solving.

Step 3 **Carry out the plan.** Once you know how to approach the problem, carry out your plan. You may run into "dead ends" and unforeseen roadblocks, but be persistent.

Step 4 **Look back and check.** Check your answer to see that it is reasonable. Does it satisfy the conditions of the problem? Have you answered all the questions the problem asks? Can you solve the problem a different way and come up with the same answer?

In Step 2 of Polya's problem-solving method, we are told to devise a plan. Here are some strategies that may prove useful.

Problem-Solving Strategies

Make a table or a chart.	If a formula applies, use it.
Look for a pattern.	Work backward.
Solve a similar simpler problem.	Guess and check.
Draw a sketch.	Use trial and error.
Use inductive reasoning.	Use common sense.
Write an equation and solve it.	Look for a "catch" if an answer seems too obvious or impossible.

A particular problem solution may involve one or more of the strategies listed here, and you should try to be creative in your problem-solving techniques. The examples that follow illustrate some of these strategies.

Using a Table or Chart

Fibonacci (1170–1250) discovered the sequence named after him in a problem on rabbits. Fibonacci (son of Bonaccio) is one of several names for Leonardo of Pisa. His father managed a warehouse in present-day Bougie (or Bejaia), in Algeria. Thus it was that Leonardo Pisano studied with a Moorish teacher and learned the "Indian" numbers that the Moors and other Moslems brought with them in their westward drive.

Fibonacci wrote books on algebra, geometry, and trigonometry.

EXAMPLE 1 Solving Fibonacci's Rabbit Problem

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will there be at the end of 1 year? (This problem is a famous one in the history of mathematics and first appeared in *Liber Abaci,* a book written by the Italian mathematician Leonardo Pisano (also known as Fibonacci) in the year 1202.)

SOLUTION

We apply Polya's method.

Step 1 **Understand the problem.** After several readings, we can reword the problem as follows:

How many pairs of rabbits will the man have at the end of one year if he starts with one pair, and they reproduce this way: During the first month of life, each pair produces no new rabbits, but each month thereafter each pair produces one new pair?

Step 2 **Devise a plan.** Because there is a definite pattern to how the rabbits will reproduce, we can construct Table 2. Once the table is completed, the final entry in the final column is our answer.

TABLE 2

Month	Number of Pairs at Start	Number of New Pairs Produced	Number of Pairs at End of Month
1^{st}			
2^{nd}			
3^{rd}			
4^{th}			
5^{th}			
6^{th}			
7^{th}			
8^{th}			
9^{th}			
10^{th}			
11^{th}			
12^{th}			

The answer will go here.

Step 3 **Carry out the plan.** At the start of the first month, there is only one pair of rabbits. No new pairs are produced during the first month, so there is $1 + 0 = 1$ pair present at the end of the first month. This pattern continues. In the table, we add the number in the first column of numbers to the number in the second column to get the number in the third.

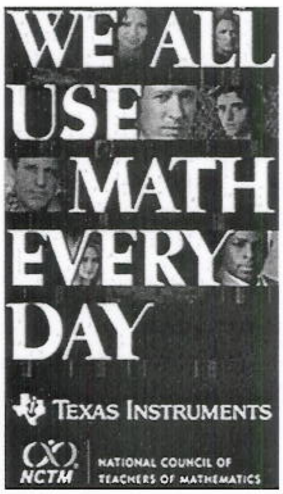
Month	Number of Pairs at Start	+	Number of New Pairs Produced	=	Number of Pairs at End of Month	
1^{st}	1		0		1	1 + 0 = 1
2^{nd}	1		1		2	1 + 1 = 2
3^{rd}	2		1		3	2 + 1 = 3
4^{th}	3		2		5	.
5^{th}	5		3		8	.
6^{th}	8		5		13	.
7^{th}	13		8		21	.
8^{th}	21		13		34	.
9^{th}	34		21		55	.
10^{th}	55		34		89	.
11^{th}	89		55		144	.
12^{th}	144		89		233	144 + 89 = 233

The answer is the final entry.

There will be 233 pairs of rabbits at the end of one year.

Step 4 **Look back and check.** This problem can be checked by going back and making sure that we have interpreted it correctly, which we have. Double-check the arithmetic. We have answered the question posed by the problem, so the problem is solved.

The sequence shown in color in the table in Example 1 is the Fibonacci sequence, mentioned in Example 1 of the previous section. In the remaining examples of this section, we use Polya's process but do not list the steps specifically as we did in Example 1.



On January 23, 2005, the CBS television network presented the first episode of *NUMB3RS*, a show focusing on how mathematics is used in solving crimes. David Krumholtz plays Charlie Eppes, a brilliant mathematician who assists his FBI agent brother (Rob Morrow). In the first-season episode "Sabotage" (2/25/2005), one of the agents admits that she "never saw how math relates to the real world," and Charlie uses the **Fibonacci sequence** and its relationship to nature to enlighten her.

Working Backward

EXAMPLE 2 Determining a Wager at the Track

Rob Zwettler goes to the racetrack with his buddies on a weekly basis. One week he tripled his money, but then lost $12. He took his money back the next week, doubled it, but then lost $40. The following week he tried again, taking his money back with him. He quadrupled it, and then played well enough to take that much home with him, a total of $224. How much did he start with the first week?

SOLUTION

This problem asks us to find Rob's starting amount, given information about his winnings and losses. We also know his final amount. The method of working backward can be applied quite easily.

Because his final amount was $224 and this represents four times the amount he started with on the third week, we *divide* $224 by 4 to find that he started the third week with $56. Before he lost $40 the second week, he had this $56 plus the $40 he lost, giving him $96. This represented double what he started with, so he started with $96 *divided by* 2, or $48, the second week. Repeating this process once more for the first week, before his $12 loss he had

$$\$48 + \$12 = \$60,$$

Augustus De Morgan was an English mathematician and philosopher, who served as professor at the University of London. He wrote numerous books, one of which was *A Budget of Paradoxes.* His work in set theory and logic led to laws that bear his name and are covered in other chapters. He died in the same year as Charles Babbage.

which represents triple what he started with. Therefore, he started with

$$\$60 \div 3 = \$20. \quad \text{Answer}$$

To check, observe the following equations that depict winnings and losses:

First week: $(3 \times \$20) - \$12 = \$60 - \$12 = \$48$
Second week: $(2 \times \$48) - \$40 = \$96 - \$40 = \$56$
Third week: $(4 \times \$56) = \$224.$ His final amount

Using Trial and Error Recall that $5^2 = 5 \cdot 5 = 25$, that is, 5 squared is 25. Thus, 25 is called a **perfect square.** Other perfect squares include

$$1, \quad 4, \quad 9, \quad 16, \quad 36, \quad \text{and so on.} \quad \text{Perfect squares}$$

The next example uses the idea of perfect square.

EXAMPLE 3 Finding Augustus De Morgan's Birth Year

The mathematician Augustus De Morgan lived in the nineteenth century. He made the following statement: "I was x years old in the year x^2." In what year was he born?

SOLUTION

We must find the year of De Morgan's birth. The problem tells us that he lived in the nineteenth century, which is another way of saying that he lived during the 1800s. One year of his life was a perfect square, so we must find a number between 1800 and 1900 that is a perfect square. Use trial and error.

$$42^2 = 1764$$
$$43^2 = 1849 \quad \text{1849 is between 1800 and 1900.}$$
$$44^2 = 1936$$

The only natural number whose square is between 1800 and 1900 is 43, since $43^2 = 1849$. Therefore, De Morgan was 43 years old in 1849. The final step in solving the problem is to subtract 43 from 1849 to find the year of his birth:

$$1849 - 43 = 1806. \quad \text{He was born in 1806.}$$

Although the following suggestion for a check may seem unorthodox, it works: Look up De Morgan's birth date in a book dealing with mathematics history, such as *An Introduction to the History of Mathematics*, Sixth Edition, by Howard W. Eves.

Guessing and Checking As mentioned above, $5^2 = 25$. The inverse (opposite) of squaring a number is called taking the **square root**. We indicate the positive square root using a **radical sign** $\sqrt{\ }$. Thus, $\sqrt{25} = 5$. Also,

$$\sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{16} = 4, \quad \text{and so on.} \quad \text{Square roots}$$

The next problem deals with a square root, and dates back to Hindu mathematics, circa 850.

EXAMPLE 4 Finding the Number of Camels

One-fourth of a herd of camels was seen in the forest; twice the square root of that herd had gone to the mountain slopes; and 3 times 5 camels remained on the riverbank. What is the numerical measure of that herd of camels?

SOLUTION

The numerical measure of a herd of camels must be a counting number. Because the problem mentions "one-fourth of a herd" and "the square root of that herd," the number of camels must be both a multiple of 4 and a perfect square, so that only whole numbers are used. The least counting number that satisfies both conditions is 4. We write an equation where x represents the numerical measure of the herd, and then substitute 4 for x to see if it is a solution.

$$\underbrace{\frac{1}{4}x}_{\text{One-fourth of the herd}} + \underbrace{2\sqrt{x}}_{\text{Twice the square root of that herd}} + \underbrace{3 \cdot 5}_{\text{3 times 5 camels}} = \underbrace{x}_{\text{The numerical measure of the herd.}}$$

$$\frac{1}{4}(4) + 2\sqrt{4} + 3 \cdot 5 = 4 \quad \text{Let } x = 4.$$

$$1 + 4 + 15 = 4 \quad ? \quad \sqrt{4} = 2$$

$$20 \neq 4$$

Because 4 is not the solution, try 16, the next perfect square that is a multiple of 4.

$$\frac{1}{4}(16) + 2\sqrt{16} + 3 \cdot 5 = 16 \quad \text{Let } x = 16.$$

$$4 + 8 + 15 = 16 \quad ? \quad \sqrt{16} = 4$$

$$27 \neq 16$$

Because 16 is not a solution, try 36.

$$\frac{1}{4}(36) + 2\sqrt{36} + 3 \cdot 5 = 36 \quad \text{Let } x = 36.$$

$$9 + 12 + 15 = 36 \quad ? \quad \sqrt{36} = 6$$

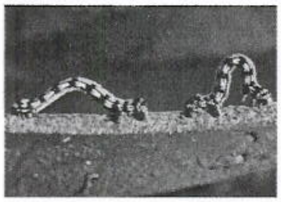
$$36 = 36$$

We see that 36 is the numerical measure of the herd. Check in the words of the problem: "One-fourth of 36, plus twice the square root of 36, plus 3 times 5" gives 9 plus 12 plus 15, which equals 36. (Algebra shows that 36 is the *only* correct answer.)

Considering a Similar Simpler Problem

EXAMPLE 5 Finding the Units Digit of a Power

The digit farthest to the right in a counting number is called the *ones* or *units* digit, because it tells how many ones are contained in the number when grouping by tens is considered. What is the ones (or units) digit in 2^{4000}?



The 1952 film *Hans Christian Andersen* features Danny Kaye as the Danish writer of fairy tales. In a scene outside a schoolhouse, he sings a song to an inchworm: "Inchworm, inchworm, measuring the marigolds, you and your arithmetic, you'll probably go far." Following the scene, students in the schoolhouse are heard singing arithmetic facts:

Two and two are four,
Four and four are eight,
Eight and eight are sixteen,
Sixteen and sixteen are thirty-two.

Their answers are all **powers of 2.**

SOLUTION

Recall that 2^{4000} means that 2 is used as a factor 4000 times:

$$2^{4000} = \underbrace{2 \times 2 \times 2 \times \ldots \times 2.}_{4000 \text{ factors}}$$

Certainly, we are not expected to evaluate this number. To answer the question, we examine some smaller powers of 2 and then look for a pattern. We start with the exponent 1 and look at the first twelve powers of 2.

$$\begin{array}{lll} 2^1 = 2 & 2^5 = 32 & 2^9 = 512 \\ 2^2 = 4 & 2^6 = 64 & 2^{10} = 1024 \\ 2^3 = 8 & 2^7 = 128 & 2^{11} = 2048 \\ 2^4 = 16 & 2^8 = 256 & 2^{12} = 4096 \end{array}$$

Notice that in each of the four rows above, the ones digit is the same. The final row, which contains the exponents 4, 8, and 12, has the ones digit 6. Each of these exponents is divisible by 4, and because 4000 is divisible by 4, we can use inductive reasoning to predict that the units digit in 2^{4000} is 6.

(*Note*: The units digit for any other power can be found if we divide the exponent by 4 and consider the remainder. Then compare the result to the list of powers above. For example, to find the units digit of 2^{543}, divide 543 by 4 to get a quotient of 135 and a remainder of 3. The units digit is the same as that of 2^3, which is 8.)

Drawing a Sketch

EXAMPLE 6 Connecting the Dots

An array of nine dots is arranged in a 3×3 square, as shown in Figure 6. Is it possible to join the dots with exactly four straight line segments if you are not allowed to pick up your pencil from the paper and may not trace over a segment that has already been drawn? If so, show how.

FIGURE 6

SOLUTION

Figure 7 shows three attempts. In each case, something is wrong. In the first sketch, one dot is not joined. In the second, the figure cannot be drawn without picking up your pencil from the paper or tracing over a line that has already been drawn. In the third figure, all dots have been joined, but you have used five line segments as well as retraced over the figure.

FIGURE 7

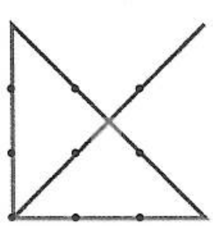


FIGURE 8

The conditions of the problem can be satisfied, as shown in Figure 8. We "went outside of the box," which was not prohibited by the conditions of the problem. This is an example of creative thinking—we used a strategy that is usually not considered at first.

In *Die Hard: With a Vengeance* (see the Chapter Opener), Simon taunts McClane with a riddle that has its origins in Egyptian mathematics.

As I was going to St. Ives,
I met a man with seven wives.
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kittens.
Kittens, cats, sacks and wives,
How many were going to St. Ives?

"My phone number is 555 and the answer. Call me in 30 seconds or die."

By calling 555-0001, he was able to contact Simon. Do you see why 1 is the answer to this riddle? (Use **common sense.**)

Using Common Sense The final example falls into a category of problems that involve a "catch." Some of these problems seem too easy or perhaps impossible at first because we tend to overlook an obvious situation. Look carefully at the use of language in such problems. And, of course, never forget to use common sense.

EXAMPLE 7 Determining Coin Denominations

Two currently minted United States coins together have a total value of $1.05. One is not a dollar. What are the two coins?

SOLUTION

Our initial reaction might be, "The only way to have two such coins with a total of $1.05 is to have a nickel and a dollar, but the problem says that one of them is not a dollar." This statement is indeed true. What we must realize here is that the one that is not a dollar is the nickel, and the *other* coin is a dollar! So the two coins are a dollar and a nickel.

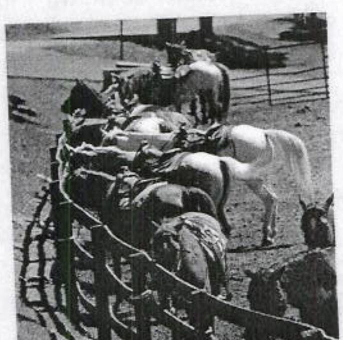
Solution to the Chapter Opener Problem This is one way to do it: With both jugs empty, fill the 3-gallon jug and pour its contents into the 5-gallon jug. Then fill the 3-gallon jug again, and pour it into the 5-gallon jug until the latter is filled. There is now $(3 + 3) - 5 = 1$ gallon in the 3-gallon jug. Empty the 5-gallon jug, and pour the 1 gallon of water from the 3-gallon jug into the 5-gallon jug. Finally, fill the 3-gallon jug and pour all of it into the 5-gallon jug, resulting in $1 + 3 = 4$ gallons in the 5-gallon jug.

(*Note*: There is another way to solve this problem. See if you can discover the alternative solution.)

For Further Thought

A Brain Teaser

Various forms of the following problem have been around for many years.



In Farmer Jack's will, Jack bequeathed $\frac{1}{2}$ of his horses to his son Johnny, $\frac{1}{3}$ to his daughter Linda, and $\frac{1}{9}$ to his son Jeff. Jack had 17 horses, so how were they to comply with the terms of the will? Certainly, horses cannot be divided up into fractions. Their attorney, Garbarino, came to their rescue, and was able to execute the will to the satisfaction of all. How did she do it?

Here is the solution:

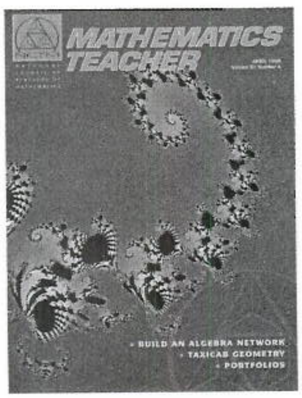
Garbarino added one of her horses to the 17, giving a total of 18. Johnny received $\frac{1}{2}$ of 18, or 9, Linda received $\frac{1}{3}$ of 18, or 6, and Jeff received $\frac{1}{9}$ of 18, or 2. That accounted for a total of $9 + 6 + 2 = 17$ horses. Then Garbarino took back her horse, and everyone was happy.

For Group Discussion or Individual Investigation

Make up a similar problem involving fractions. Check your work.

1.3 Exercises

One of the most popular features in the journal Mathematics Teacher, *published by the National Council of Teachers of Mathematics, is the monthly calendar, which provides an interesting, unusual, or challenging problem for each day of the month. Problems are contributed by the editors of the journal, teachers, and students, and the contributors are cited in each issue. Exercises 1–35 are problems chosen from these calendars over the past several years, with the day, month, and year for the problem indicated. The authors want to thank the many contributors for permission to use these problems.*



Use the various problem-solving strategies to solve each problem. In many cases there is more than one possible approach, so be creative.

1. ***Catwoman's Cats*** If you ask Batman's nemesis, Catwoman, how many cats she has, she answers with a riddle: "Five-sixths of my cats plus seven." How many cats does Catwoman have? (April 20, 2003) 42

2. ***Pencil Collection*** Bob gave four-fifths of his pencils to Barbara, then he gave two-thirds of the remaining pencils to Bonnie. If he ended up with ten pencils for himself, with how many did he start? (October 12, 2003) 150

3. ***Adding Gasoline*** The gasoline gauge on a van initially read $\frac{1}{8}$ full. When 15 gallons were added to the tank, the gauge read $\frac{3}{4}$ full. How many more gallons are needed to fill the tank? (November 25, 2004) 6

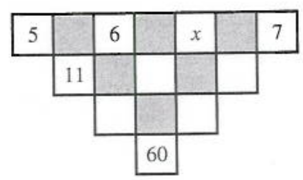
4. ***Gasoline Tank Capacity*** When 6 gallons of gasoline are put into a car's tank, the indicator goes from $\frac{1}{4}$ of a tank to $\frac{5}{8}$. What is the total capacity of the gasoline tank? (February 21, 2004) 16 gallons

5. ***Number Pattern*** What is the relationship between the rows of numbers?

18,	38,	24,	46,	42
8,	24,	8,	24,	8

(May 26, 2005)

6. ***Unknown Number*** The number in an unshaded square is obtained by adding the numbers connected with it from the row above. (The 11 is one such number.) What is the value of x? (August 9, 2004) 10



7. ***Locking Boxes*** You and I each have one lock and a corresponding key. I want to mail you a box with a ring in it, but any box that is not locked will be emptied before it reaches its recipient. How can I safely send you the ring? (Note that you and I each have keys to our own lock but not to the other lock.) (May 4, 2004)

8. ***Woodchuck Chucking Wood*** Nine woodchucks can chuck eight pieces of wood in 3 hours. How much wood can a woodchuck chuck in 1 hour? (May 24, 2004) $\frac{8}{27}$ of a piece of wood

9. ***Number in a Sequence*** In the sequence 16, 80, 48, 64, A, B, C, D, each term beyond the second term is the arithmetic mean (average) of the two previous terms. What is the value of D? (April 26, 2004) 59

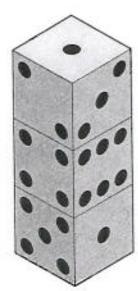
10. ***Unknown Number*** Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been if she had worked the problem correctly? (September 3, 2004) 15

11. ***Labeling Boxes*** You are working in a store that has been very careless with the stock. Three boxes of socks are each incorrectly labeled. The labels say *red socks, green socks*, and *red and green socks*. How can you relabel the boxes correctly by taking only one sock out of one box, without looking inside the boxes? (October 22, 2001)

12. ***Vertical Symmetry in States' Names*** (If a vertical line is drawn through the center of a figure and the left and right sides are reflections of each other across this line, the figure is said to have vertical symmetry.) When spelled with all capital letters, each letter in HAWAII has vertical symmetry. Find the name of a state whose letters all have vertical and horizontal symmetry. (September 11, 2001) OHIO

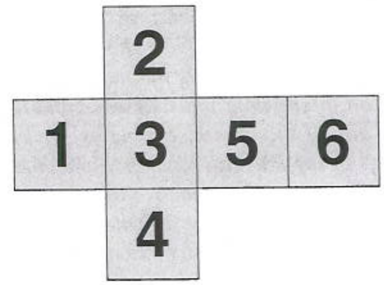
13. ***Sum of Hidden Dots on Dice*** Three dice with faces numbered 1 through 6 are stacked as shown. Seven of the eighteen faces are visible, leaving eleven faces hidden on the back, on the bottom, and between dice. The total number of dots not visible in this view is __D__.

A. 21
B. 22
C. 31
D. 41
E. 53
(September 17, 2001)

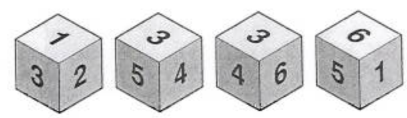


14. ***Mr. Green's Age*** At his birthday party, Mr. Green would not directly tell how old he was. He said, "If you add the year of my birth to this year, subtract the year of my tenth birthday and the year of my fiftieth birthday, and then add my present age, the result is eighty." How old was Mr Green? (December 14, 1997) 70

15. ***Unfolding and Folding a Box*** An unfolded box is shown below.



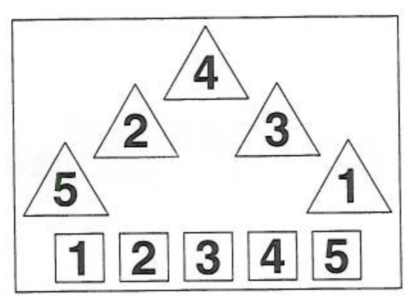
Which figure shows the box folded up? (November 7, 2001) A



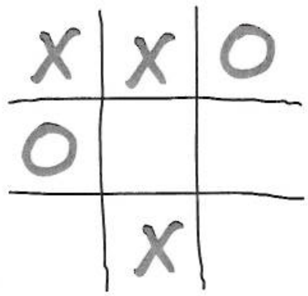
A B C D

16. ***Age of the Bus Driver*** Today is your first day driving a city bus. When you leave downtown, you have twenty-three passengers. At the first stop, three people exit and five people get on the bus. At the second stop, eleven people exit and eight people get on the bus. At the third stop, five people exit and ten people get on. How old is the bus driver? (April 1, 2002)

17. ***Matching Triangles and Squares*** How can you connect each square with the triangle that has the same number? Lines cannot cross, enter a square or triangle, or go outside the diagram. (October 15, 1999)



18. ***Ticktacktoe Strategy*** You and a friend are playing ticktacktoe, where three in a row loses. (See the next page.) You are O. If you want to win, what must your next move be? (October 21, 2001)



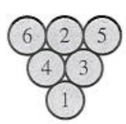
19. ***Forming Perfect Square Sums*** How must one place the integers from 1 to 15 in each of the spaces below in such a way that no number is repeated and the sum of the numbers in any two consecutive spaces is a perfect square? (November 11, 2001)

9	7	2	14	11	5	4	12	13	3	6	10	15	1	8

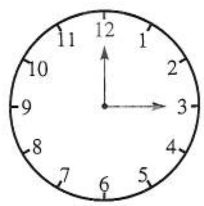
(or the same arrangement reading right to left)

20. ***How Old?*** Pat and Chris have the same birthday. Pat is twice as old as Chris was when Pat was as old as Chris is now. If Pat is now 24 years old, how old is Chris? (December 3, 2001) 18

21. ***Difference Triangle*** Balls numbered 1 through 6 are arranged in a *difference triangle*. Note that in any row, the difference between the larger and the smaller of two successive balls is the number of the ball that appears below them. Arrange balls numbered 1 through 10 in a *difference triangle*. (May 6, 1998)



22. ***Clock Face*** By drawing two straight lines, divide the face of a clock into three regions such that the numbers in the regions have the same total. (October 28, 1998)



23. ***Alphametric*** If a, b, and c are digits for which

$$\begin{array}{r} 7\ a\ 2 \\ -4\ 8\ b \\ \hline c\ 7\ 3, \end{array}$$

then $a + b + c =$ __D__.

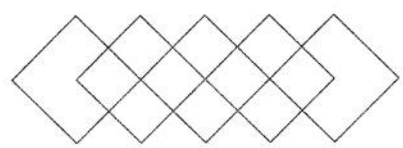
A. 14 **B.** 15 **C.** 16 **D.** 17 **E.** 18
(September 22, 1999)

24. ***Perfect Square*** Only one of these numbers is a perfect square. Which one is it? (October 8, 1997) 329476

329476 389372 964328
326047 724203

25. ***Sleeping on the Way to Grandma's House*** While traveling to his grandmother's for Christmas, George fell asleep halfway through the journey. When he awoke, he still had to travel half the distance that he had traveled while sleeping. For what part of the entire journey had he been asleep? (December 25, 1998) $\frac{1}{3}$

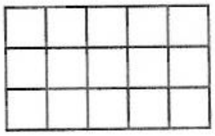
26. ***Counting Puzzle (Rectangles)*** How many rectangles of any size are in the figure shown? (September 10, 2001) 31



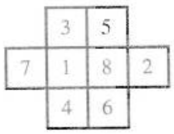
27. ***Buckets of Water*** You have brought two unmarked buckets to a stream. The buckets hold 7 gallons and 3 gallons of water, respectively. How can you obtain exactly 5 gallons of water to take home? (October 19, 1997)

28. ***Collecting Acorns*** Chipper and Dalie collected thirty-two acorns on Monday and stored them with their acorn supply. After Chipper fell asleep, Dalie ate half the acorns. This pattern continued through Friday night, with thirty-two acorns being added and half the supply being eaten. On Saturday morning, Chipper counted the acorns and found that they had only thirty-five. How many acorns had they started with on Monday morning? (March 12, 1997) 128

29. *Counting Puzzle (Rectangles)* How many rectangles are in the figure? (March 27, 1997) 90



30. *Digit Puzzle* Place each of the digits 1, 2, 3, 4, 5, 6, 7, and 8 in separate boxes so that boxes that share common corners do not contain successive digits. (November 29, 1997) Here is one possible solution.



31. *Palindromic Number* (*Note:* A *palindromic* number is a number whose digits read the same left to right as right to left. For example, 383, 12321, and 9876789 are palindromic.) The odometer of the family car read 15951 when the driver noticed that the number was palindromic. "Curious," said the driver to herself. "It will be a long time before that happens again." But 2 hours later, the odometer showed a new palindromic number. (*Author's note:* Assume it was the next possible one.) How fast was the car driving in those 2 hours? (December 26, 1998) 55 miles per hour

32. *Exchange Rate* An island has no currency; it instead has the following exchange rate:

$$50 \text{ bananas} = 20 \text{ coconuts}$$
$$30 \text{ coconuts} = 12 \text{ fish}$$
$$100 \text{ fish} = 1 \text{ hammock}$$

How many bananas equal 1 hammock? (April 16, 1998) 625 bananas

33. *Final Digits of a Power of 7* What are the final two digits of 7^{1997}? (November 29, 1997) 07

34. *Brightness of a Clock Display* If a digital clock is the only light in an otherwise totally dark room, when will the room be darkest? Brightest? (May 1, 1996)

35. *Value of Coins* Which is worth more, a kilogram of \$10 gold pieces or half a kilogram of \$20 gold pieces? (March 20, 1995)

36. *Units Digit of a Power of 3* If you raise 3 to the 324th power, what is the units digit of the result? 1

37. *Units Digit of a Power of 7* What is the units digit in 7^{491}? 3

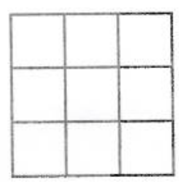
38. *Money Spent at a Bazaar* Ashley O'Shaughnessy bought a book for \$10 and then spent half her remaining money on a train ticket. She then bought lunch for \$4 and spent half her remaining money at a bazaar. She left the bazaar with \$8. How much money did she start with? \$50

39. *Unknown Number* I am thinking of a positive number. If I square it, double the result, take half of that result, and then add 12, I get 37. What is my number? 5

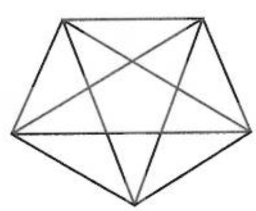
40. *Frog Climbing up a Well* A frog is at the bottom of a 20-foot well. Each day it crawls up 4 feet, but each night it slips back 3 feet. After how many days will the frog reach the top of the well? 17 days

41. *Matching Socks* A drawer contains 20 black socks and 20 white socks. If the light is off and you reach into the drawer to get your socks, what is the minimum number of socks you must pull out in order to be sure that you have a matching pair? 3 socks

42. *Counting Puzzle (Squares)* How many squares are in the following figure? 14



43. *Counting Puzzle (Triangles)* How many triangles are in the following figure? 35



44. ***Children in a Circle*** Some children are standing in a circular arrangement. They are evenly spaced and marked in numerical order. The fourth child is standing directly opposite the twelfth child. How many children are there in the circle? 16

45. ***Perfect Number*** A *perfect number* is a counting number that is equal to the sum of all its counting number divisors except itself. For example, 28 is a perfect number because its divisors other than itself are 1, 2, 4, 7, and 14, and $1 + 2 + 4 + 7 + 14 = 28$. What is the least perfect number? 6

46. ***Naming Children*** Becky's mother has three daughters. She named her first daughter Penny and her second daughter Nichole. What did she name her third daughter? Becky

47. ***Growth of a Lily Pad*** A lily pad grows so that each day it doubles its size. On the twentieth day of its life, it completely covers a pond. On what day was the pond half covered? the nineteenth day

48. ***Interesting Property of a Sentence*** Comment on an interesting property of this sentence: "A man, a plan, a canal, Panama." (*Hint:* See Exercise 31.)

49. ***High School Graduation Year of Author*** One of the authors of this book graduated from high school in the year that satisfies these conditions: (1) The sum of the digits is 23; (2) The hundreds digit is 3 more than the tens digit; (3) No digit is an 8. In what year did he graduate? 1967

50. ***Relative Heights*** Donna is taller than David but shorter than Bill. Dan is shorter than Bob. What is the first letter in the name of the tallest person?

51. ***Adam and Eve's Assets*** Eve said to Adam, "If you give me one dollar, then we will have the same amount of money." Adam then replied, "Eve, if you give me one dollar, I will have double the amount of money you are left with." How much does each have? Eve has \$5, and Adam has \$7.

52. ***Missing Digits Puzzle*** In the addition problem at the top of the next column, some digits are missing as indicated by the blanks. If the problem is done correctly, what is the sum of the missing digits? The sum of the missing digits is 14.

$$\begin{array}{r} \underline{4}\ 3\ 5 \\ 8\ \underline{2}\ 6 \\ +\ 1\ 4\ \underline{7} \\ \hline \underline{1}\ 4\ 0\ 8 \end{array}$$

53. ***Missing Digits Puzzle*** Fill in the blanks so that the multiplication problem below uses all digits 0, 1, 2, 3, . . . , 9 exactly once, and is correctly worked.

$$\begin{array}{r} \underline{4}\ 0\ 2 \\ \times\quad 3\ \underline{9} \\ \hline \underline{1}\ 5,\underline{6}\ \underline{7}\ \underline{8} \end{array}$$

54. ***Magic Square*** A *magic square* is a square array of numbers that has the property that the sum of the numbers in any row, column, or diagonal is the same. Fill in the square below so that it becomes a magic square, and all digits 1, 2, 3, . . . , 9 are used exactly once.

6		8
	5	
		4

55. ***Magic Square*** Refer to Exercise 54. Complete the magic square below so that all counting numbers 1, 2, 3, . . . ,16 are used exactly once, and the sum in each row, column, or diagonal is 34.

6			9
	15		14
11		10	
16		13	

56. ***Paying for a Mint*** Brian Altobello has an unlimited number of cents (pennies), nickels, and dimes. In how many different ways can he pay 15¢ for a chocolate mint? (For example, one way is 1 dime and 5 pennies.) 6 ways

57. ***Pitches in a Baseball Game*** What is the minimum number of pitches that a baseball player who pitches a complete game can make in a regulation 9-inning baseball game?