1.1 Solving Problems by Inductive Reasoning

Characteristics of Inductive and Deductive Reasoning • Pitfalls of Inductive Reasoning

The **Moscow papyrus,** which dates back to about 1850 B.C., provides an example of **inductive reasoning** by the early Egyptian mathematicians. Problem 14 in the document reads:

You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

What does all this mean? A *frustum* of a pyramid is that part of the pyramid remaining after its top has been cut off by a plane parallel to the base of the pyramid. The formula for finding the volume of the frustum of a pyramid with a square base is

$$V = \frac{1}{3}h(b^2 + bB + B^2),$$

where b is the area of the upper base, B is the area of the lower base, and h is the height (or altitude). The writer of the problem is giving a method of determining the volume of the frustum of a pyramid with square bases on the top and bottom, with bottom base side of length 4, top base side of length 2, and height equal to 6.

A truncated pyramid, or frustum of a pyramid

Characteristics of Inductive and Deductive Reasoning

The development of mathematics can be traced to the Egyptian and Babylonian cultures (3000 B.C.–A.D. 260) as a necessity for problem solving. To solve a problem or perform an operation, a cookbook-like recipe was given, and it was performed repeatedly to solve similar problems. During the classical Greek period (600 B.C.–A.D. 450), general concepts were applied to specific problems, resulting in a structured, logical development of mathematics.

By observing that a specific method worked for a certain type of problem, the Babylonians and the Egyptians concluded that the same method would work for any similar type of problem. Such a conclusion is called a *conjecture*. A **conjecture** is an educated guess based on repeated observations of a particular process or pattern. The method of reasoning we have just described is called *inductive reasoning*.

Inductive Reasoning

Inductive reasoning is characterized by drawing a general conclusion (making a conjecture) from repeated observations of specific examples. The conjecture may or may not be true.

In testing a conjecture obtained by inductive reasoning, it takes only one example that does not work in order to prove the conjecture false. Such an example is called a **counterexample.**

Inductive reasoning provides a powerful method of drawing conclusions, but there is no assurance that the observed conjecture will always be true. For this reason, mathematicians are reluctant to accept a conjecture as an absolute truth until it is formally proved using methods of *deductive reasoning*. Deductive reasoning characterized the development and approach of Greek mathematics, as seen in the works of Euclid, Pythagoras, Archimedes, and others.

Deductive Reasoning

Deductive reasoning is characterized by applying general principles to specific examples.

We now look at examples of these two types of reasoning. In this chapter, we often refer to the **natural** or **counting numbers:**

$$1, 2, 3, \ldots .$$ Natural (counting) numbers

(↑ Ellipsis points)

The three dots (*ellipsis points*) indicate that the numbers continue indefinitely in the pattern that has been established. The most probable rule for continuing this pattern is "add 1 to the previous number," and this is indeed the rule that we follow.

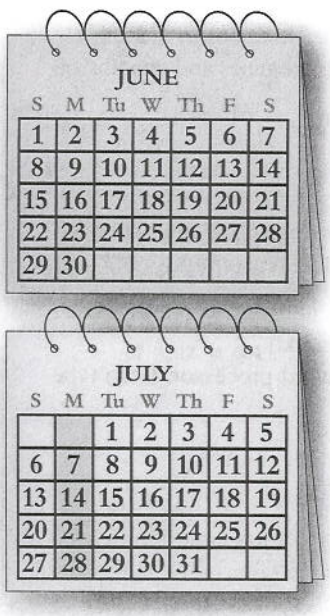


FIGURE 1

Now consider the following list of natural numbers: 2, 9, 16, 23, 30. What is the next number of this list? What is the pattern? After studying the numbers, we might see that $2 + 7 = 9$, and $9 + 7 = 16$. Do we add 16 and 7 to get 23? Do we add 23 and 7 to get 30? Yes; any number in the given list can be found by adding 7 to the preceding number, so the next number in the list should be $30 + 7 = 37$.

We set out to find the "next number" by reasoning from observation of the numbers in the list. We may have jumped from these observations to the general statement that any number in the list is 7 more than the preceding number. This is an example of *inductive reasoning*.

By using inductive reasoning, we concluded that 37 was the next number in the list. But this is wrong. We've been tricked into drawing an incorrect conclusion. The person making up the list has another answer in mind. The list of numbers

$$2, 9, 16, 23, 30$$

actually gives the dates of Mondays in June if June 1 falls on a Sunday. The next Monday after June 30 is July 7. With this pattern, the list continues as

$$2, 9, 16, 23, 30, 7, 14, 21, 28, \ldots.$$

See the calendar in Figure 1. The process used to obtain the rule "add 7" in the preceding list reveals one main flaw of inductive reasoning. We can never be sure that what is true in a specific case will be true in general. Inductive reasoning does not guarantee a true result, but it does provide a means of making a conjecture.

Throughout this book, we use *exponents* to represent repeated multiplication. For example, in the expression 4^3 the exponent is 3:

$$4^3 = 4 \cdot 4 \cdot 4 = 64. \quad \text{4 is used as a factor 3 times.}$$

Exponential Expression

If a is a number and n is a counting number $(1, 2, 3, \ldots)$, then the exponential expression a^n is defined as

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}.$$

The number a is the *base* and n is the exponent.

With deductive reasoning, we use general statements and apply them to specific situations. For example, one of the best-known rules in mathematics is the Pythagorean theorem: In any right triangle, the sum of the squares of the legs (shorter sides) is equal to the square of the hypotenuse (longest side). Thus, if we know that the lengths of the shorter sides are 3 inches and 4 inches, we can find the length of the longest side. Let h represent the longest side.

$$3^2 + 4^2 = h^2 \quad \text{Pythagorean theorem}$$
$$9 + 16 = h^2 \quad 3^2 = 3 \cdot 3 = 9;\ 4^2 = 4 \cdot 4 = 16$$
$$25 = h^2 \quad \text{Add.}$$
$$5 = h \quad \text{The positive square root of 25 is 5.}$$

Thus, the longest side measures 5 inches. We used the general rule (the Pythagorean theorem) and applied it to the specific situation.

Reasoning through a problem usually requires certain *premises*. A **premise** can be an assumption, law, rule, widely held idea, or observation. Then reason inductively or deductively from the premises to obtain a **conclusion.** The premises and conclusion make up a **logical argument.**

EXAMPLE 1 Identifying Premises and Conclusions

Identify each premise and the conclusion in each of the following arguments. Then tell whether each argument is an example of inductive or deductive reasoning.

(a) Our house is made of brick. Both of my next-door neighbors have brick houses. Therefore, all houses in our neighborhood are made of brick.
(b) All word processors will type the symbol @. I have a word processor. I can type the symbol @.
(c) Today is Monday. Tomorrow will be Tuesday.

SOLUTION

(a) The premises are "Our house is made of brick" and "Both of my next-door neighbors have brick houses." The conclusion is "Therefore, all houses in our neighborhood are made of brick." Because the reasoning goes from specific examples to a general statement, the argument is an example of inductive reasoning (although it may very well have a false conclusion).
(b) Here, the premises are "All word processors will type the symbol @" and "I have a word processor." The conclusion is "I can type the symbol @." This reasoning goes from general to specific, so deductive reasoning was used.
(c) There is only one premise here, "Today is Monday." The conclusion is "Tomorrow will be Tuesday." The fact that Tuesday immediately follows Monday is being used, even though this fact is not explicitly stated. Because the conclusion comes from general facts that apply to this special case, deductive reasoning was used.

The earlier calendar example illustrated how inductive reasoning may, at times, lead to false conclusions. However, in many cases, inductive reasoning does provide correct results if we look for the most *probable* answer.

In the 2003 movie *A Wrinkle in Time*, young Calvin O'Keefe, played by Gregory Smith, is challenged to identify a particular sequence of numbers. He correctly identifies it as the **Fibonacci sequence.**

EXAMPLE 2 Predicting the Next Number in a Sequence

Use inductive reasoning to determine the *probable* next number in each list below.

(a) 5, 9, 13, 17, 21, 25 **(b)** 1, 1, 2, 3, 5, 8, 13, 21 **(c)** 2, 4, 8, 16, 32

SOLUTION

(a) Each number in the list is obtained by adding 4 to the previous number. The probable next number is $25 + 4 = 29$.
(b) Beginning with the third number in the list, 2, each number is obtained by adding the two previous numbers in the list. That is,

$$1 + 1 = 2, \qquad 1 + 2 = 3, \qquad 2 + 3 = 5,$$

and so on. The probable next number in the list is $13 + 21 = 34$. (These are the first few terms of the famous *Fibonacci sequence*.)

(c) It appears here that to obtain each number after the first, we must double the previous number. Therefore, the most probable next number is $32 \times 2 = 64$.

Inductive reasoning often can be used to predict an answer in a list of similarly constructed computation exercises, as shown in the next example.

EXAMPLE 3 Predicting the Product of Two Numbers

$37 \times 3 = 111$
$37 \times 6 = 222$
$37 \times 9 = 333$
$37 \times 12 = 444$

Consider the list of equations in the margin. Use the list to predict the next multiplication fact in the list.

SOLUTION

In each case, the left side of the equation has two factors, the first 37 and the second a multiple of 3, beginning with 3. The product (answer) in each case consists of three digits, all the same, beginning with 111 for 37×3. For this pattern to continue, the next multiplication fact would be $37 \times 15 = 555$, which is indeed true. (*Note:* You might want to investigate what occurs after 30 is reached for the right-hand factor, and make conjectures based on those products.)

TABLE 1

Number of Points	Number of Regions
1	1
2	2
3	4
4	8
5	16

Pitfalls of Inductive Reasoning There are pitfalls associated with inductive reasoning. A classic example involves the maximum number of regions formed when chords are constructed in a circle. When two points on a circle are joined with a line segment, a *chord* is formed. Locate a single point on a circle. Because no chords are formed, a single interior region is formed. See Figure 2(a). Locate two points and draw a chord. Two interior regions are formed, as shown in Figure 2(b). Continue this pattern. Locate three points, and draw all possible chords. Four interior regions are formed, as shown in Figure 2(c). Four points yield 8 regions and five points yield 16 regions. See Figures 2(d) and 2(e).

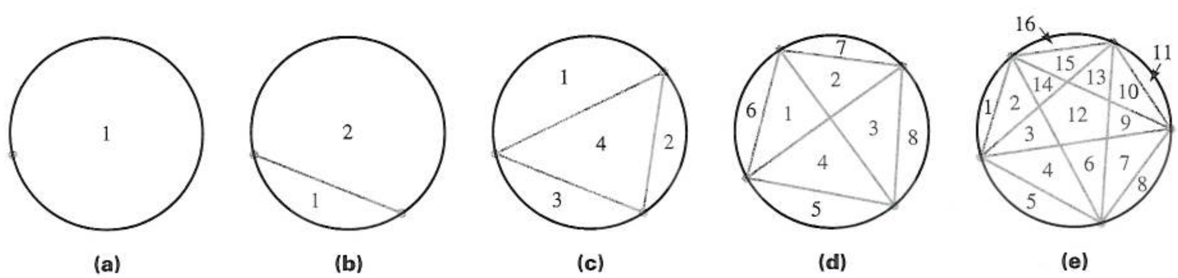


FIGURE 2

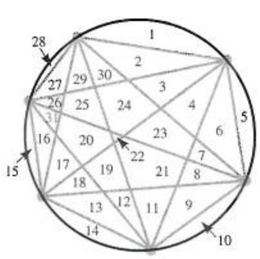


FIGURE 3

The results of the preceding observations are summarized in Table 1 in the margin. The pattern formed in the column headed "Number of Regions" is the same one we saw in Example 2(c), where we predicted that the next number would be 64. It seems here that for each additional point on the circle, the number of regions doubles. A reasonable inductive conjecture would be that for six points, 32 regions would be formed. But as Figure 3 indicates, there are only 31 regions! The pattern of doubling ends when the sixth point is considered. Adding a seventh point would yield 57 regions. The numbers obtained here are

$$1, 2, 4, 8, 16, 31, 57.$$

For n points on the circle, the number of regions is given by the formula

$$\frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.^*$$

We can use a graphing calculator to construct a table of values that indicates the number of regions for various numbers of points. Using X rather than n, we can define Y_1 using the expression above (see Figure 4(a) on the next page). Then, creating a table of values, as in Figure 4(b), we see how many regions (indicated by Y_1) there are for any number of points (X).

As indicated earlier, not until a general relationship is proved can one be sure about a conjecture because one counterexample is always sufficient to make the conjecture false.

For Further Thought

Inductive Reasoning Anecdote

The following anecdote concerning inductive reasoning appears in the first volume of the *In Mathematical Circles* series by Howard Eves (PWS-KENT Publishing Company).

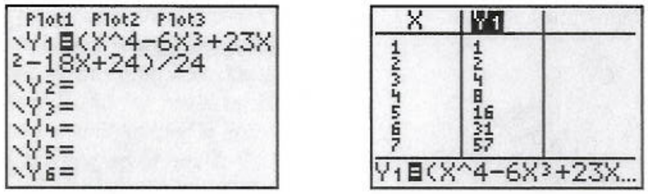
A scientist had two large jars before him on the laboratory table. The jar on his left contained 100 fleas; the jar on his right was empty. The scientist carefully lifted a flea from the jar on the left, placed the flea on the table between the two jars, stepped back, and in a loud voice said, "Jump." The flea jumped and was put in the jar on the right. A second flea was carefully lifted from the jar on the left and placed on the table between the two jars. Again the scientist stepped back and in a loud voice said, "Jump." The flea jumped and was put in the jar on the right. In the same manner, the scientist treated each of the 100 fleas in the jar on the left, and each flea jumped as ordered. The two jars were then interchanged and the experiment continued with a slight difference. This time the scientist carefully lifted a flea from the jar on the left, yanked off its hind legs, placed the flea on the table between the jars, stepped back, and in a loud voice said, "Jump." The flea did not jump, and was put in the jar on the right. A second flea was carefully lifted from the jar on the left, its hind legs yanked off, and then placed on the table between the two jars. Again the scientist stepped back and in a loud voice said, "Jump." The flea did not jump, and was put in the jar on the right. In this manner, the scientist treated each of the 100 fleas in the jar on the left, and in no case did a flea jump when ordered. So the scientist recorded in his notebook the following induction: "A flea, if its hind legs are yanked off, cannot hear."

For Group Discussion or Individual Investigation

Discuss or research examples from advertising on television, in newspapers, magazines, etc., that lead consumers to draw incorrect conclusions.

*For more information on this and other similar patterns, see "Counting Pizza Pieces and Other Combinatorial Problems," by Eugene Maier, in the January 1988 issue of *Mathematics Teacher*, pp. 22–26.

Note the careful use of parentheses.



(a) (b)

FIGURE 4

1.1 EXERCISES

In Exercises 1–12, determine whether the reasoning is an example of deductive or inductive reasoning.

1. If the mechanic says that it will take seven days to repair your car, then it will actually take ten days. The mechanic says, "I figure it'll take a week to fix it, ma'am." Then you can expect it to be ready ten days from now. deductive

2. If you take your medicine, you'll feel a lot better. You take your medicine. Therefore, you'll feel a lot better. deductive

3. It has rained every day for the past nine days, and it is raining today as well. So it will also rain tomorrow. inductive

4. Marin's first three children were boys. If she has another baby, it will be a boy. inductive

5. Finley had 95 baseball cards. His mom gave him 20 more for his birthday. Therefore, he now has 115 of them. deductive

6. If the same number is subtracted from both sides of a true equation, the new equation is also true. I know that $9 + 18 = 27$. Therefore, $(9 + 18) - 12 = 27 - 12$. deductive

7. If you build it, they will come. You build it. Therefore, they will come. deductive

8. All men are mortal. Socrates is a man. Therefore, Socrates is mortal. deductive

9. It is a fact that every student who ever attended Geekville University was accepted into graduate school. Because I am attending Geekville, I can expect to be accepted to graduate school, too. inductive

10. For the past 53 years, a rare plant has bloomed in Columbia each summer, alternating between yellow and green flowers. Last summer, it bloomed with green flowers, so this summer it will bloom with yellow flowers. inductive

11. In the sequence 5, 10, 15, 20, . . . , the most probable next number is 25. inductive

12. Carrie Underwood's last four single releases have reached the Top Ten country list, so her current release will also reach the Top Ten. inductive

13. Discuss the differences between inductive and deductive reasoning. Give an example of each. Answers will vary.

14. Give an example of faulty inductive reasoning. Answers will vary.

Determine the most probable next term in each list of numbers.

15. 6, 9, 12, 15, 18 21

16. 13, 18, 23, 28, 33 38

17. 3, 12, 48, 192, 768 3072

18. 32, 16, 8, 4, 2 1

19. 3, 6, 9, 15, 24, 39 63

20. $\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}$ $\frac{11}{13}$

21. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$ $\frac{11}{12}$

22. 1, 4, 9, 16, 25 36

23. 1, 8, 27, 64, 125 216

24. 2, 6, 12, 20, 30, 42 56

25. 4, 7, 12, 19, 28, 39 52

26. −1, 2, −3, 4, −5, 6 −7

27. 5, 3, 5, 5, 3, 5, 5, 5, 3, 5, 5, 5, 5, 3, 5, 5, 5, 5 5

28. 8, 2, 8, 2, 2, 8, 2, 2, 2, 8, 2, 2, 2, 2, 8, 2, 2, 2, 2 2

29. Construct a list of numbers similar to those in Exercise 15 such that the most probable next number in the list is 60. One such list is 10, 20, 30, 40, 50, ...

30. Construct a list of numbers similar to those in Exercise 26 such that the most probable next number in the list is 9. One such list is 1, −2, 3, −4, 5, −6, 7, −8, ...

In Exercises 31–42, a list of equations is given. Use the list and inductive reasoning to predict the next equation, and then verify your conjecture.

31.
$$(9 \times 9) + 7 = 88$$
$$(98 \times 9) + 6 = 888$$
$$(987 \times 9) + 5 = 8888$$
$$(9876 \times 9) + 4 = 88{,}888$$
$$(98{,}765 \times 9) + 3 = 888{,}888$$

32.
$$(1 \times 9) + 2 = 11$$
$$(12 \times 9) + 3 = 111$$
$$(123 \times 9) + 4 = 1111$$
$$(1234 \times 9) + 5 = 11{,}111$$
$$(12{,}345 \times 9) + 6 = 111{,}111$$

33.
$$3367 \times 3 = 10{,}101$$
$$3367 \times 6 = 20{,}202$$
$$3367 \times 9 = 30{,}303$$
$$3367 \times 12 = 40{,}404$$
$$3367 \times 15 = 50{,}505$$

34.
$$15873 \times 7 = 111{,}111$$
$$15873 \times 14 = 222{,}222$$
$$15873 \times 21 = 333{,}333$$
$$15873 \times 28 = 444{,}444$$
$$15{,}873 \times 35 = 555{,}555$$

35.
$$34 \times 34 = 1156$$
$$334 \times 334 = 111{,}556$$
$$3334 \times 3334 = 11{,}115{,}556$$
$$33{,}334 \times 33{,}334 = 1{,}111{,}155{,}556$$

36.
$$11 \times 11 = 121$$
$$111 \times 111 = 12{,}321$$
$$1111 \times 1111 = 1{,}234{,}321$$
$$11{,}111 \times 11{,}111 = 123{,}454{,}321$$

37.
$$3 = \frac{3(2)}{2}$$
$$3 + 6 = \frac{6(3)}{2}$$
$$3 + 6 + 9 = \frac{9(4)}{2}$$
$$3 + 6 + 9 + 12 = \frac{12(5)}{2}$$
$$3 + 6 + 9 + 12 + 15 = \frac{15(6)}{2}$$

38.
$$2 = 4 - 2$$
$$2 + 4 = 8 - 2$$
$$2 + 4 + 8 = 16 - 2$$
$$2 + 4 + 8 + 16 = 32 - 2$$
$$2 + 4 + 8 + 16 + 32 = 64 - 2$$

39.
$$5(6) = 6(6 - 1)$$
$$5(6) + 5(36) = 6(36 - 1)$$
$$5(6) + 5(36) + 5(216) = 6(216 - 1)$$
$$5(6) + 5(36) + 5(216) + 5(1296) = 6(1296 - 1)$$

40.
$$3 = \frac{3(3 - 1)}{2}$$
$$3 + 9 = \frac{3(9 - 1)}{2}$$
$$3 + 9 + 27 = \frac{3(27 - 1)}{2}$$
$$3 + 9 + 27 + 81 = \frac{3(81 - 1)}{2}$$

41.
$$\frac{1}{2} = 1 - \frac{1}{2}$$
$$\frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1 - \frac{1}{16}$$

42.
$$\frac{1}{1 \cdot 2} = \frac{1}{2}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

A story is often told about how the great mathematician Carl Friedrich Gauss (1777–1855) at a very young age was told by his teacher to find the sum of the first 100 counting numbers. While his classmates toiled at the problem, Carl simply wrote down a single number and handed it in to his teacher. His answer was correct. When asked how he did it, the young Carl explained that he observed that there were 50 pairs of numbers that each added up to 101. (See below.) So the sum of all the numbers must be 50 × 101 = 5050.

$1 + 2 + 3 + \cdots + 98 + 99 + 100$

101

101

101

50 sums of 101 = 50 × 101 = 5050

Use the method of Gauss to find each sum.

43. $1 + 2 + 3 + \cdots + 200$ 20,100

44. $1 + 2 + 3 + \cdots + 400$ 80,200

45. $1 + 2 + 3 + \cdots + 800$ 320,400

46. $1 + 2 + 3 + \cdots + 2000$ 2,001,000

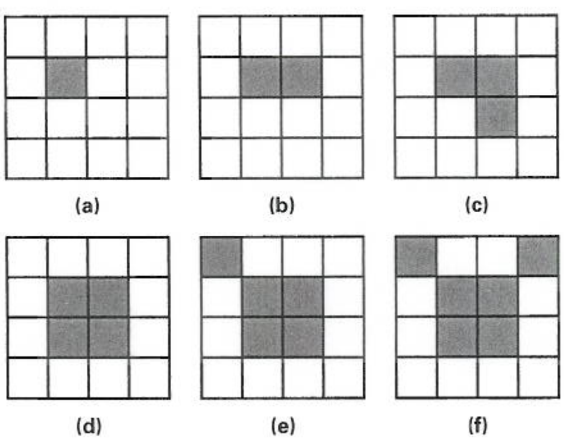
47. Modify the procedure of Gauss to find the sum $1 + 2 + 3 + \cdots + 175$. 15,400

48. Explain in your own words how the procedure of Gauss can be modified to find the sum $1 + 2 + 3 + \cdots + n$, where n is an odd natural number. (When an odd natural number is divided by 2, it leaves a remainder of 1.) Answers will vary.

49. Modify the procedure of Gauss to find the sum $2 + 4 + 6 + \cdots + 100$. 2550

50. Use the result of Exercise 49 to find the sum $4 + 8 + 12 + \cdots + 200$. 5100

51. Find a pattern in the following figures and use inductive reasoning to predict the next figure.



52. Consider the following table.

0	2	2	2	0	0	0	0	0
0	2	4	6	4	2	0	0	0
0	2	6	12	14	12	6	2	0
0	2	8	20	32	38	32	20	8

Find a pattern and predict the next row of the table.

53. What is the most probable next number in this list? 12, 1, 1, 1, 2, 1, 3 (*Hint:* Think about a clock.)

54. What is the next term in this list? O, T, T, F, F, S, S, E, N, T (*Hint:* Think about words and their relationship to numbers.)

55. **(a)** Choose any three-digit number with all different digits. Now reverse the digits, and subtract the smaller from the larger. Record your result. Choose another three-digit number and repeat this process. Do this as many times as it takes for you to see a pattern in the different results you obtain. (*Hint:* What is the middle digit? What is the sum of the first and third digits?)

(b) Write an explanation of this pattern. You may want to use this exercise as a "number trick" to amuse your friends.

56. Choose any number, and follow these steps.
(a) Multiply by 2.
(b) Add 6.
(c) Divide by 2.
(d) Subtract the number you started with.
(e) Record your result.

Repeat the process, except in Step (b), add 8. Record your final result. Repeat the process once more, except in Step (b), add 10. Record your final result.

(f) Observe what you have done; use inductive reasoning to explain how to predict the final result. You may want to use this exercise as a "number trick" to amuse your friends.

57. Complete the following.

142,857 × 1 = ______
142,857 × 2 = ______
142,857 × 3 = ______
142,857 × 4 = ______
142,857 × 5 = ______
142,857 × 6 = ______

What pattern exists in the successive answers? Now multiply 142,857 by 7 to obtain an interesting result.

58. Complete the following.

$$12{,}345{,}679 \times 9 = \underline{111{,}111{,}111}$$
$$12{,}345{,}679 \times 18 = \underline{222{,}222{,}222}$$
$$12{,}345{,}679 \times 27 = \underline{333{,}333{,}333}$$

By what number would you have to multiply 12,345,679 to get an answer of 888,888,888? 72

59. Refer to Figures 2(b)–(e) and 3. Instead of counting interior regions of the circle, count the chords formed. Use inductive reasoning to predict the number of chords that would be formed if seven points were used. 21

60. The following number trick can be performed on one of your friends. It was provided by Dr. George DeRise of Thomas Nelson Community College.

(a) Ask your friend to write down his or her age. (Only whole numbers are allowed.)

(b) Multiply the number by 4.

(c) Add 10.

(d) Multiply by 25.

(e) Subtract the number of days in a non-leap year.

(f) Add the amount of change (less than a dollar, in cents) in his or her pocket.

(g) Ask your friend for the final answer.

If you add 115 to the answer, the first two digits are the friend's age, and the last two give the amount of change.

61. Explain how a toddler might use inductive reasoning to decide on something that will be of benefit to him or her. Answers will vary.

62. Discuss one example of inductive reasoning that you have used recently in your life. Test your premises and your conjecture. Did your conclusion ultimately prove to be true or false? Answers will vary.

1.2 An Application of Inductive Reasoning: Number Patterns

Number Sequences • Successive Differences • Number Patterns and Sum Formulas • Figurate Numbers

Number Sequences

An ordered list of numbers such as

$$3, 9, 15, 21, 27, \ldots,$$

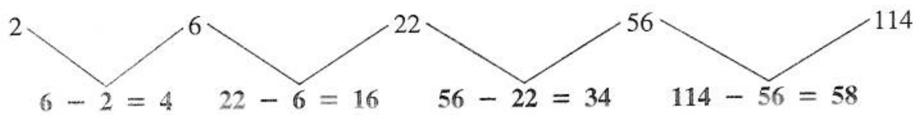
is called a *sequence*. A **number sequence** is a list of numbers having a first number, a second number, a third number, and so on, called the **terms** of the sequence. The sequences in Examples 2(a) and 2(c) in the previous section are called *arithmetic* and *geometric sequences,* respectively. An **arithmetic sequence** has a common *difference* between successive terms, while a **geometric sequence** has a common *ratio* between successive terms.

Successive Differences

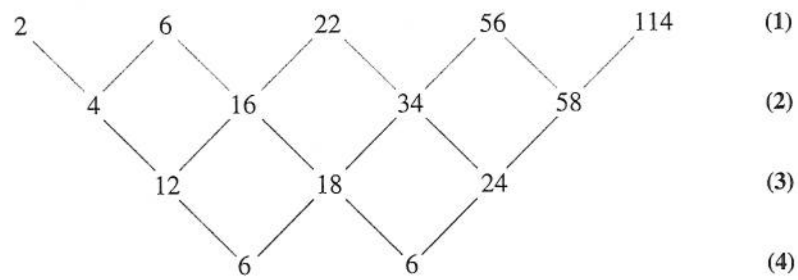
The sequences seen in the previous section were usually simple enough for us to make an obvious conjecture about the next term. However, some sequences may provide more difficulty in making such a conjecture, and often the **method of successive differences** may be applied to determine the next term if it is not obvious at first glance. Consider the sequence

$$2, 6, 22, 56, 114, \ldots.$$

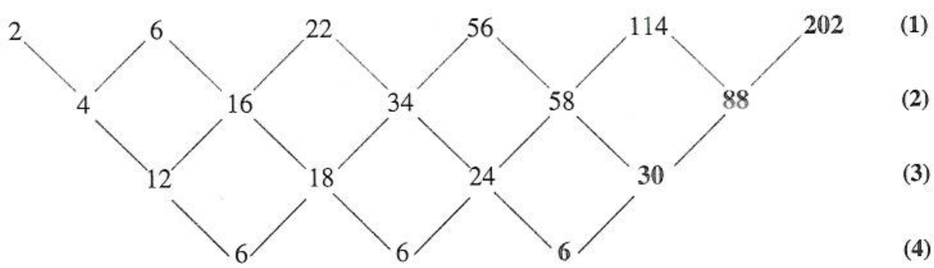
Because the next term is not obvious, subtract the first term from the second term, the second from the third, the third from the fourth, and so on.



Now repeat the process with the sequence 4, 16, 34, 58 and continue repeating until the difference is a constant value, as shown in line (4):



Once a line of constant values is obtained, simply work "backward" by adding until the desired term of the given sequence is obtained. Thus, for this pattern to continue, another 6 should appear in line (4), meaning that the next term in line (3) would have to be $24 + 6 = 30$. The next term in line (2) would be $58 + 30 = 88$. Finally, the next term in the given sequence would be $114 + 88 = $ **202**. The final scheme of numbers is shown below.



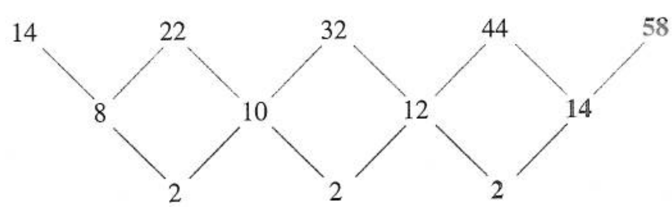
EXAMPLE 1 Using Successive Differences

Use the method of successive differences to determine the next number in each sequence.

(a) 14, 22, 32, 44, . . . **(b)** 5, 15, 37, 77, 141, . . .

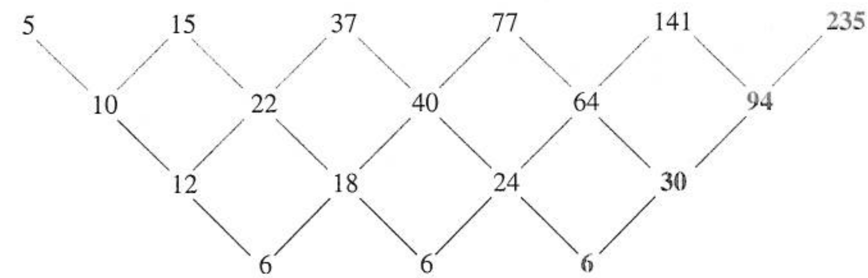
SOLUTION

(a) Using the scheme described above, obtain the following:



Once the row of 2s was obtained and extended, we were able to get $12 + 2 = 14$, and $44 + 14 = 58$, as shown above. The next number in the sequence is **58**.

(b) Proceeding as before, obtain the following diagram.



The next number in the sequence is 235.

The method of successive differences will not always work. For example, try it on the Fibonacci sequence in Example 2(b) of Section 1.1 and see what happens!

Number Patterns and Sum Formulas Mathematics features a seemingly endless variety of number patterns. Observe the following pattern:

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 5^2.$$

In each case, the left side of the equation is the indicated sum of the consecutive odd counting numbers beginning with 1, and the right side is the square of the number of terms on the left side. You should verify this in each case. Inductive reasoning would suggest that the next line in this pattern is

$$1 + 3 + 5 + 7 + 9 + 11 = 6^2.$$

Evaluating each side shows that each side simplifies to 36.

We cannot conclude that this pattern will continue indefinitely, because observation of a finite number of examples does not guarantee that the pattern will continue. However, mathematicians have proved that this pattern does indeed continue indefinitely, using a method of proof called *mathematical induction*. (See any standard college algebra text.)

Any even counting number may be written in the form $2k$, where k is a counting number. It follows that the kth odd counting number is written $2k - 1$. For example, the third odd counting number, 5, can be written $2(3) - 1$. Using these ideas, we can write the result obtained above as follows.

Sum of the First *n* Odd Counting Numbers

If n is any counting number, then

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

EXAMPLE 2 Predicting the Next Equation in a List

In each of the following, several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

(a)

$$1^2 = 1^3$$
$$(1 + 2)^2 = 1^3 + 2^3$$
$$(1 + 2 + 3)^2 = 1^3 + 2^3 + 3^3$$
$$(1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3$$

(b)

$$1 = 1^3$$
$$3 + 5 = 2^3$$
$$7 + 9 + 11 = 3^3$$
$$13 + 15 + 17 + 19 = 4^3$$

(c)

$$1 = \frac{1 \cdot 2}{2}$$
$$1 + 2 = \frac{2 \cdot 3}{2}$$
$$1 + 2 + 3 = \frac{3 \cdot 4}{2}$$
$$1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2}$$

SOLUTION

(a) The left side of each equation is the square of the sum of the first n counting numbers, while the right side is the sum of their cubes. The next equation in the pattern would be

$$(1 + 2 + 3 + 4 + 5)^2 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Each side simplifies to 225, so the pattern is true for this equation.

(b) The left sides of the equations contain the sum of odd counting numbers, starting with the first (1) in the first equation, the second and third (3 and 5) in the second equation, the fourth, fifth, and sixth (7, 9, and 11) in the third equation, and so on. The right side contains the cube (third power) of the number of terms on the left side in each case. Following this pattern, the next equation would be

$$21 + 23 + 25 + 27 + 29 = 5^3,$$

which can be verified by computation.

(c) The left side of each equation gives the indicated sum of the first n counting numbers, and the right side is always of the form

$$\frac{n(n + 1)}{2}.$$

For the pattern to continue, the next equation would be

$$1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}.$$

Because each side simplifies to 15, the pattern is true for this equation. ■

The patterns established in Examples 2(a) and 2(c) can be written as follows.

Special Sum Formulas

For any counting number n,

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

and

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The second formula given is a generalization of the method first explained preceding Exercise 43 in the previous section, relating the story of young Carl Gauss. We can provide a general, deductive argument showing how this equation is obtained. Suppose that we let S represent the sum $1 + 2 + 3 + \cdots + n$. This sum can also be written as $S = n + (n-1) + (n-2) + \cdots + 1$. Now write these two equations as follows.

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + n \\ S &= n + (n-1) + (n-2) + \cdots + 1 \\ \hline 2S &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) && \text{Add the corresponding sides.} \\ 2S &= n(n+1) && \text{There are } n \text{ terms of } n+1. \\ S &= \frac{n(n+1)}{2} && \text{Divide both sides by 2.} \end{aligned}$$

We can now apply deductive reasoning to find the sum of the first n counting numbers for any given value of n.

In the 1959 Disney animation *Donald in Mathmagic Land*, Donald Duck travels back in time to meet the Greek mathematician **Pythagoras** (c. 540 B.C.), who with his fellow mathematicians formed the Pythagorean brotherhood. The brotherhood devoted its time to the study of mathematics and music.

Figurate Numbers Pythagoras and his Pythagorean brotherhood (see the margin note) studied numbers of geometric arrangements of points, such as *triangular numbers*, *square numbers*, and *pentagonal numbers*. Figure 5 illustrates the first few of each of these types of numbers.

The *figurate numbers* possess numerous interesting patterns. Every square number greater than 1 is the sum of two consecutive triangular numbers. (For example, $9 = 3 + 6$ and $25 = 10 + 15$.) Every pentagonal number can be represented as the sum of a square number and a triangular number. (For example, $5 = 4 + 1$ and $12 = 9 + 3$.)

In the expression T_n, n is called a **subscript.** T_n is read **"T sub *n*,"** and it represents the triangular number in the nth position in the sequence. For example,

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad \text{and} \quad T_4 = 10.$$

S_n and P_n represent the nth square and pentagonal numbers, respectively.

Formulas for Triangular, Square, and Pentagonal Numbers

For any natural number n,

the nth triangular number is given by $T_n = \frac{n(n+1)}{2}$,

the nth square number is given by $S_n = n^2$, and

the nth pentagonal number is given by $P_n = \frac{n(3n-1)}{2}$.

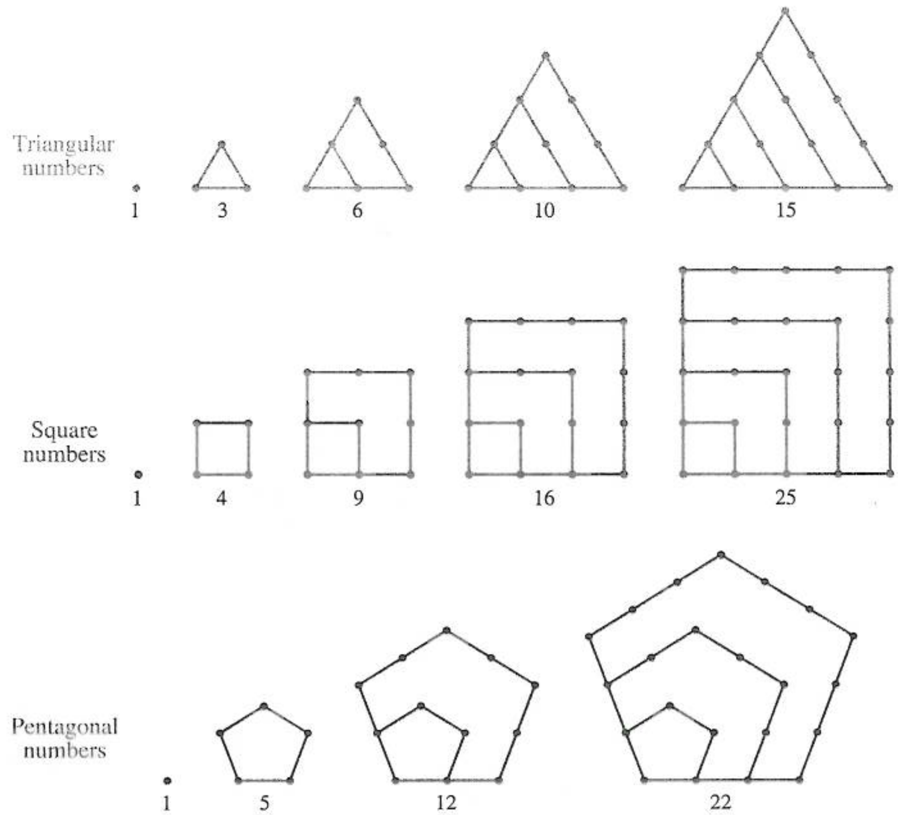


FIGURE 5

EXAMPLE 3 Using the Formulas for Figurate Numbers

Use the formulas to find each of the following.

(a) seventh triangular number
(b) twelfth square number
(c) sixth pentagonal number

SOLUTION

(a) $T_7 = \frac{n(n+1)}{2} = \frac{7(7+1)}{2} = \frac{7(8)}{2} = \frac{56}{2} = 28$ Formula for a triangular number, $n = 7$

(b) $S_{12} = n^2 = 12^2 = 144$ Formula for a square number, $n = 12$

$12^2 = 12 \cdot 12$

(c) $P_6 = \frac{n(3n-1)}{2} = \frac{6[3(6)-1]}{2} = \frac{6(18-1)}{2} = \frac{6(17)}{2} = 51$

Inside the brackets, multiply first and then subtract.

EXAMPLE 4 Illustrating a Figurate Number Relationship

Show that the sixth pentagonal number is equal to 3 times the fifth triangular number, plus 6.

SOLUTION

From Example 3(c), $P_6 = \mathbf{51}$. The fifth triangular number is 15. Thus,

$$51 = 3(15) + 6 = 45 + 6 = 51.$$

The general relationship examined in Example 4 can be written as follows.

$$P_n = 3 \cdot T_{n-1} + n \quad (n \geq 2)$$

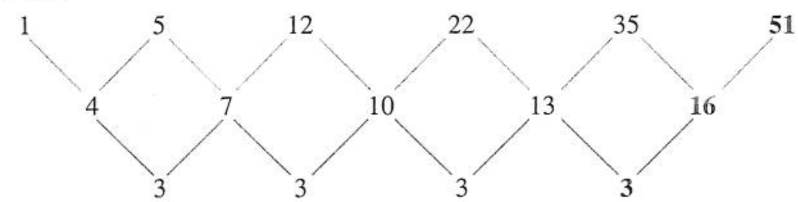
EXAMPLE 5 Predicting a Pentagonal Number

The first five pentagonal numbers are

1, 5, 12, 22, 35.

Use the method of successive differences to predict the sixth pentagonal number.

SOLUTION



After the second line of successive differences, we work backward to find that the sixth pentagonal number is **51**, which was also found in Example 3(c).

For Further Thought

Kaprekar Numbers

Take any three-digit number whose digits are not all the same. Arrange the digits in decreasing order, and then arrange them in increasing order. Now subtract. Repeat the process, using a 0 if necessary in the event that the difference consists of only two digits. For example, suppose that we choose a number whose digits are 1, 4, and 8, such as 841.

841	963	954
−148	−369	−459
693	594	495

Notice that we have obtained the number 495, and the process will lead to 495 again. The number 495 is called a **Kaprekar number.** The number 495 will eventually always be generated if this process is applied to such a three-digit number.

For Group Discussion or Individual Investigation

1. Apply the process of Kaprekar to a two-digit number, in which the digits are not the same. (Interpret 9 as 09 if necessary.) Compare the results. What seems to be true?
2. Repeat the process for four digits, comparing results after several steps. What conjecture can be made for this situation?

1.2 EXERCISES

Use the method of successive differences to determine the next number in each sequence.

1. 1, 4, 11, 22, 37, 56, . . . 79
2. 3, 14, 31, 54, 83, 118, . . . 159
3. 6, 20, 50, 102, 182, 296, . . . 450
4. 1, 11, 35, 79, 149, 251, . . . 391
5. 0, 12, 72, 240, 600, 1260, 2352, . . . 4032
6. 2, 57, 220, 575, 1230, 2317, . . . 3992
7. 5, 34, 243, 1022, 3121, 7770, 16799, . . . 32,758
8. 3, 19, 165, 771, 2503, 6483, 14409, . . . 28,675
9. Refer to Figures 2 and 3 in Section 1.1. The method of successive differences can be applied to the sequence of interior regions,

$$1, 2, 4, 8, 16, 31,$$

to find the number of regions determined by seven points on the circle. What is the next term in this sequence? How many regions would be determined by eight points? Verify this using the formula given at the end of that section. 57; 99

10. Suppose that the expression $n^2 + 3n + 1$ determines the nth term in a sequence. That is, to find the first term, let $n = 1$; to find the second term, let $n = 2$, and so on.
    (a) Find the first four terms of the sequence.
    (b) Use the method of successive differences to predict the fifth term of the sequence.
    (c) Find the fifth term by letting $n = 5$ in the expression $n^2 + 3n + 1$. Does your result agree with the one you found in part (b)?

In Exercises 11–20, several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

11. $(1 \times 9) - 1 = 8$
$(21 \times 9) - 1 = 188$
$(321 \times 9) - 1 = 2888$
$(4321 \times 9) - 1 = 38{,}888$

12. $(1 \times 8) + 1 = 9$
$(12 \times 8) + 2 = 98$
$(123 \times 8) + 3 = 987$
$(1234 \times 8) + 4 = 9876$

13. $999{,}999 \times 2 = 1{,}999{,}998$
$999{,}999 \times 3 = 2{,}999{,}997$
$999{,}999 \times 4 = 3{,}999{,}996$

14. $101 \times 101 = 10{,}201$
$10{,}101 \times 10{,}101 = 102{,}030{,}201$
$1{,}010{,}101 \times 1{,}010{,}101 = 1{,}020{,}304{,}030{,}201$

15. $3^2 - 1^2 = 2^3$
$6^2 - 3^2 = 3^3$
$10^2 - 6^2 = 4^3$
$15^2 - 10^2 = 5^3$ $21^2 - 15^2 = 6^3$

16. $1 = 1^2$
$1 + 2 + 1 = 2^2$
$1 + 2 + 3 + 2 + 1 = 3^2$
$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$
$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2$

17. $2^2 - 1^2 = 2 + 1$
$3^2 - 2^2 = 3 + 2$
$4^2 - 3^2 = 4 + 3$ $5^2 - 4^2 = 5 + 4$

18. $1^2 + 1 = 2^2 - 2$
$2^2 + 2 = 3^2 - 3$
$3^2 + 3 = 4^2 - 4$ $4^2 + 4 = 5^2 - 5$

19. $1 = 1 \times 1$
$1 + 5 = 2 \times 3$
$1 + 5 + 9 = 3 \times 5$ $1 + 5 + 9 + 13 = 4 \times 7$

20. $1 + 2 = 3$
$4 + 5 + 6 = 7 + 8$
$9 + 10 + 11 + 12 = 13 + 14 + 15$
$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$

Use the formula $S = \frac{n(n+1)}{2}$ *to find each sum.*

21. $1 + 2 + 3 + \cdots + 300$ 45,150
22. $1 + 2 + 3 + \cdots + 500$ 125,250
23. $1 + 2 + 3 + \cdots + 675$ 228,150
24. $1 + 2 + 3 + \cdots + 825$ 340,725

Use the formula $S = n^2$ *to find each sum. (Hint: To find n, add 1 to the last term and divide by 2.)*

25. $1 + 3 + 5 + \cdots + 101$ 2601
26. $1 + 3 + 5 + \cdots + 49$ 625
27. $1 + 3 + 5 + \cdots + 999$ 250,000
28. $1 + 3 + 5 + \cdots + 301$ 22,801