Exercises 25–34 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

**25.** $x^2 + y^2 = 49$ Down 3, left 2

**26.** $x^2 + y^2 = 25$ Up 3, left 4

**27.** $y = x^3$ Left 1, down 1

**28.** $y = x^{2/3}$ Right 1, down 1

**29.** $y = \sqrt{x}$ Left 0.81

**30.** $y = -\sqrt{x}$ Right 3

**31.** $y = 2x - 7$ Up 7

**32.** $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

**33.** $y = 1/x$ Up 1, right 1

**34.** $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 35–54.

**35.** $y = \sqrt{x + 4}$ **36.** $y = \sqrt{9 - x}$

**37.** $y = |x - 2|$ **38.** $y = |1 - x| - 1$

**39.** $y = 1 + \sqrt{x - 1}$ **40.** $y = 1 - \sqrt{x}$

**41.** $y = (x + 1)^{2/3}$ **42.** $y = (x - 8)^{2/3}$

**43.** $y = 1 - x^{2/3}$ **44.** $y + 4 = x^{2/3}$

**45.** $y = \sqrt[3]{x - 1} - 1$ **46.** $y = (x + 2)^{3/2} + 1$

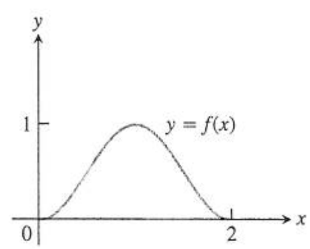
**47.** $y = \frac{1}{x - 2}$ **48.** $y = \frac{1}{x} - 2$

**49.** $y = \frac{1}{x} + 2$ **50.** $y = \frac{1}{x + 2}$

**51.** $y = \frac{1}{(x - 1)^2}$ **52.** $y = \frac{1}{x^2} - 1$

**53.** $y = \frac{1}{x^2} + 1$ **54.** $y = \frac{1}{(x + 1)^2}$

**55.** The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



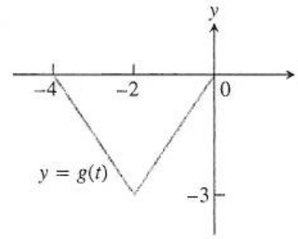

**a.** $f(x) + 2$ **b.** $f(x) - 1$

**c.** $2f(x)$ **d.** $-f(x)$

**e.** $f(x + 2)$ **f.** $f(x - 1)$

**g.** $f(-x)$ **h.** $-f(x + 1) + 1$

**56.** The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.




**a.** $g(-t)$ **b.** $-g(t)$

**c.** $g(t) + 3$ **d.** $1 - g(t)$

**e.** $g(-t + 2)$ **f.** $g(t - 2)$

**g.** $g(1 - t)$ **h.** $-g(t - 4)$

### Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

**57.** $y = x^2 - 1$, stretched vertically by a factor of 3

**58.** $y = x^2 - 1$, compressed horizontally by a factor of 2

**59.** $y = 1 + \frac{1}{x^2}$, compressed vertically by a factor of 2

**60.** $y = 1 + \frac{1}{x^2}$, stretched horizontally by a factor of 3

**61.** $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

**62.** $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

**63.** $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

**64.** $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

**65.** $y = 1 - x^3$, compressed horizontally by a factor of 3

**66.** $y = 1 - x^3$, stretched horizontally by a factor of 2

### Graphing

In Exercises 67–74, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

**67.** $y = -\sqrt{2x + 1}$ **68.** $y = \sqrt{1 - \frac{x}{2}}$

**69.** $y = (x - 1)^3 + 2$ **70.** $y = (1 - x)^3 + 2$

**71.** $y = \frac{1}{2x} - 1$ **72.** $y = \frac{2}{x^2} + 1$

**73.** $y = -\sqrt[3]{x}$ **74.** $y = (-2x)^{2/3}$

**75.** Graph the function $y = |x^2 - 1|$.

**76.** Graph the function $y = \sqrt{|x|}$.

### Ellipses

Exercises 77–82 give equations of ellipses. Put each equation in standard form and sketch the ellipse.

**77.** $9x^2 + 25y^2 = 225$ **78.** $16x^2 + 7y^2 = 112$

**79.** $3x^2 + (y - 2)^2 = 3$ **80.** $(x + 1)^2 + 2y^2 = 4$

**81.** $3(x-1)^2+2(y+2)^2=6$

**82.** $6\left(x+\frac{3}{2}\right)^2+9\left(y-\frac{1}{2}\right)^2=54$

**83.** Write an equation for the ellipse $(x^2/16)+(y^2/9)=1$ shifted 4 units to the left and 3 units up. Sketch the ellipse and identify its center and major axis.

**84.** Write an equation for the ellipse $(x^2/4)+(y^2/25)=1$ shifted 3 units to the right and 2 units down. Sketch the ellipse and identify its center and major axis.

### Combining Functions

**85.** Assume that $f$ is an even function, $g$ is an odd function, and both $f$ and $g$ are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

| | | |
|---|---|---|
| **a.** $fg$ | **b.** $f/g$ | **c.** $g/f$ |
| **d.** $f^2=ff$ | **e.** $g^2=gg$ | **f.** $f\circ g$ |
| **g.** $g\circ f$ | **h.** $f\circ f$ | **i.** $g\circ g$ |

**86.** Can a function be both even and odd? Give reasons for your answer.

T **87.** (*Continuation of Example* 1.) Graph the functions $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T **88.** Let $f(x)=x-7$ and $g(x)=x^2$. Graph $f$ and $g$ together with $f\circ g$ and $g\circ f$.

## 1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

### Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius $r$ is defined as the number of "radius units" contained in the arc $s$ subtended by that central angle. If we denote this central angle by $\theta$ when measured in radians, this means that $\theta=s/r$ (Figure 1.38), or

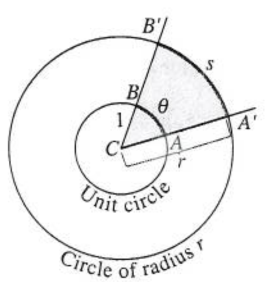



FIGURE 1.38 The radian measure of the central angle $A'CB'$ is the number $\theta=s/r$. For a unit circle of radius $r=1$, $\theta$ is the length of arc $AB$ that central angle $ACB$ cuts from the unit circle.

$$s=r\theta \qquad (\theta \text{ in radians}). \tag{1}$$

If the circle is a unit circle having radius $r=1$, then from Figure 1.38 and Equation (1), we see that the central angle $\theta$ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or $2\pi$ radians, we have

$$\pi \text{ radians} = 180° \tag{2}$$

and

$$1 \text{ radian} = \frac{180}{\pi}\ (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180}\ (\approx 0.017) \text{ radians}.$$

Table 1.2 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.2 Angles measured in degrees and radians

| Degrees | −180 | −135 | −90 | −45 | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ (radians) | $-\pi$ | $\frac{-3\pi}{4}$ | $\frac{-\pi}{2}$ | $\frac{-\pi}{4}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

An angle in the $xy$-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive $x$-axis (Figure 1.39). Angles measured counterclockwise from the positive $x$-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

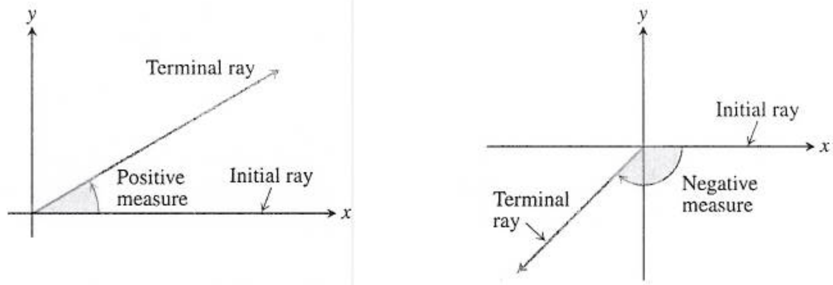



FIGURE 1.39 Angles in standard position in the $xy$-plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond $2\pi$ radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.40).

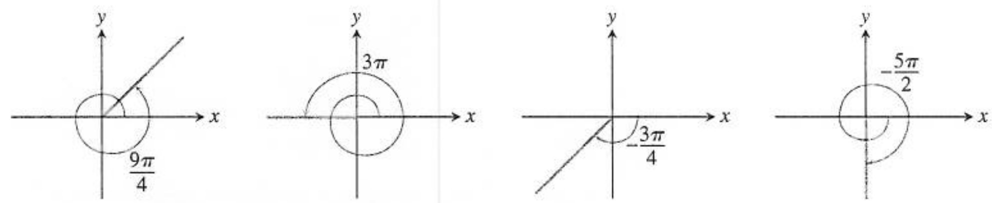
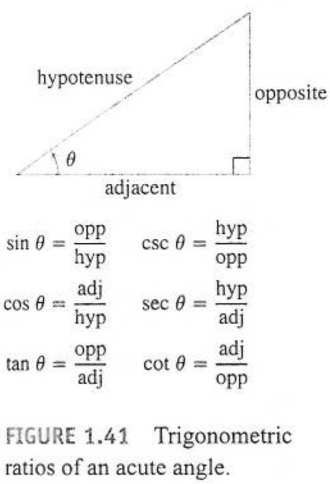



FIGURE 1.40 Nonzero radian measures can be positive or negative and can go beyond $2\pi$.

Angle Convention: Use Radians From now on, in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. We use radians because it simplifies many of the operations in calculus, and some results we will obtain involving the trigonometric functions are not true when angles are measured in degrees.

## The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.41). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius $r$. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.42).




$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.41 Trigonometric ratios of an acute angle.

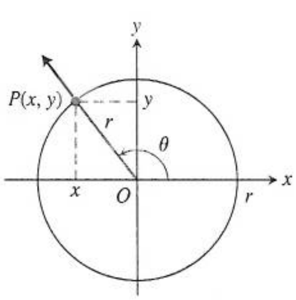



FIGURE 1.42 The trigonometric functions of a general angle $\theta$ are defined in terms of $x$, $y$, and $r$.

**sine:** $\sin\theta = \dfrac{y}{r}$ **cosecant:** $\csc\theta = \dfrac{r}{y}$

**cosine:** $\cos\theta = \dfrac{x}{r}$ **secant:** $\sec\theta = \dfrac{r}{x}$

**tangent:** $\tan\theta = \dfrac{y}{x}$ **cotangent:** $\cot\theta = \dfrac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute.

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if $\theta$ is $\pm\pi/2, \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of $\theta$ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.43. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

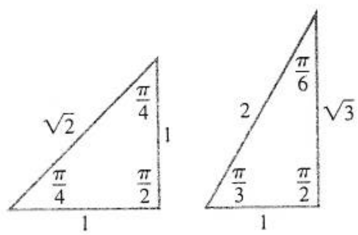



FIGURE 1.43 Radian angles and side lengths of two common triangles.

The CAST rule (Figure 1.44) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.45, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

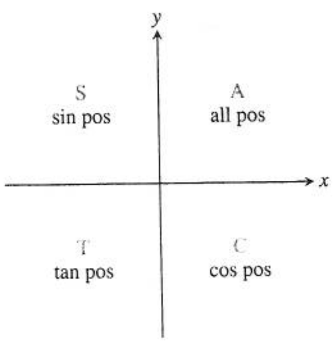



FIGURE 1.44 The CAST rule, remembered by the statement "Calculus Activates Student Thinking," tells which trigonometric functions are positive in each quadrant.

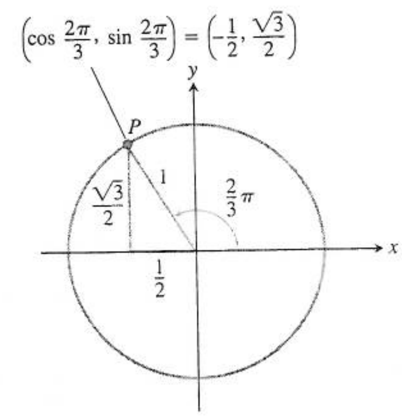



FIGURE 1.45 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we determined the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in Table 1.3.

TABLE 1.3 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of $\theta$

| **Degrees** | **−180** | **−135** | **−90** | **−45** | **0** | **30** | **45** | **60** | **90** | **120** | **135** | **150** | **180** | **270** | **360** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$\theta$ (radians)** | $-\pi$ | $\frac{-3\pi}{4}$ | $\frac{-\pi}{2}$ | $\frac{-\pi}{4}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| $\sin\theta$ | 0 | $\frac{-\sqrt{2}}{2}$ | −1 | $\frac{-\sqrt{2}}{2}$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | −1 | 0 |
| $\cos\theta$ | −1 | $\frac{-\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{-\sqrt{2}}{2}$ | $\frac{-\sqrt{3}}{2}$ | −1 | 0 | 1 |
| $\tan\theta$ | 0 | 1 | | −1 | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | | $-\sqrt{3}$ | −1 | $\frac{-\sqrt{3}}{3}$ | 0 | | 0 |

## Periodicity and Graphs of the Trigonometric Functions

When an angle of measure $\theta$ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic.*

**Periods of Trigonometric Functions**

**Period $\pi$:** $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

**Period $2\pi$:** $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number $p$ such that $f(x + p) = f(x)$ for every value of $x$. The smallest such value of $p$ is the **period** of $f$.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by $x$ instead of $\theta$. Figure 1.46 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period $2\pi$. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

**Even**

$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

**Odd**

$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

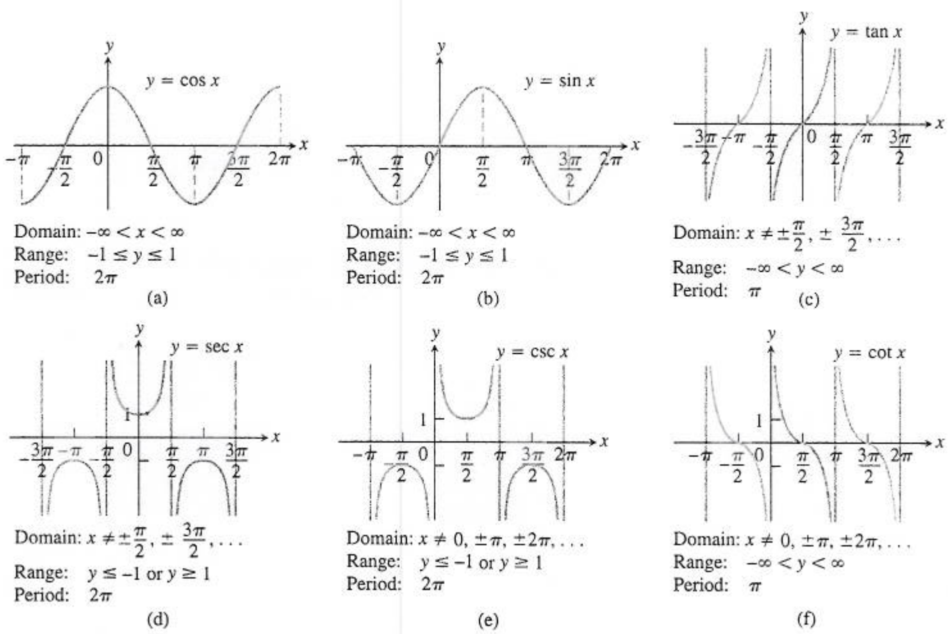



FIGURE 1.46 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

## Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance $r$ from the origin and the angle $\theta$ that ray $OP$ makes with the positive $x$-axis (Figure 1.42). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.47 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{3}$$

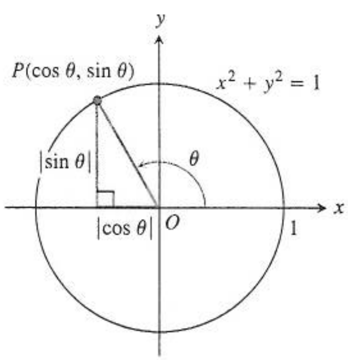



FIGURE 1.47 The reference triangle for a general angle $\theta$.

This equation, true for all values of $\theta$, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The following formulas hold for all angles $A$ and $B$ (Exercise 58).

**Addition Formulas**

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (4)$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting $\theta$ for both $A$ and $B$ in the addition formulas gives

**Double-Angle Formulas**

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad (5)$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

**Half-Angle Formulas**

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (6)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (7)$$

### The Law of Cosines

If $a$, $b$, and $c$ are sides of a triangle $ABC$ and if $\theta$ is the angle opposite $c$, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (8)$$

This equation is called the **law of cosines**.

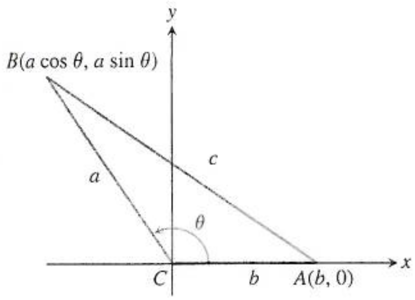



FIGURE 1.48 The square of the distance between $A$ and $B$ gives the law of cosines.

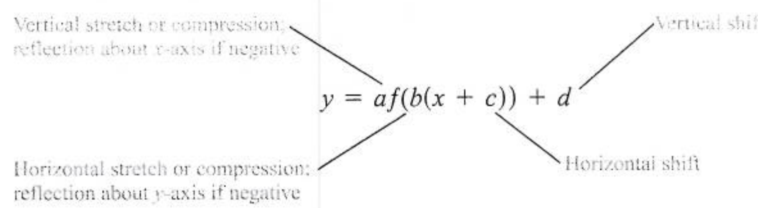
We can see why the law holds if we introduce coordinate axes with the origin at $C$ and the positive $x$-axis along one side of the triangle, as in Figure 1.48. The coordinates of $A$ are $(b, 0)$; the coordinates of $B$ are $(a\cos\theta, a\sin\theta)$. The square of the distance between $A$ and $B$ is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

## Transformations of Trigonometric Graphs

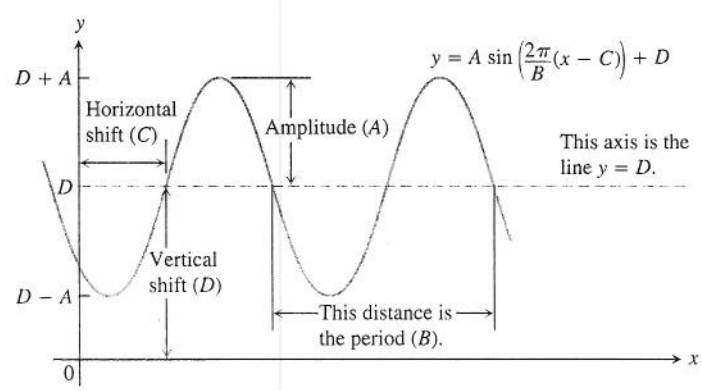
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.




The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, $C$ is the *horizontal shift*, and $D$ is the *vertical shift*. A graphical interpretation of the various terms is revealing and given below.




## Two Special Inequalities

For any angle $\theta$ measured in radians,

$$-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

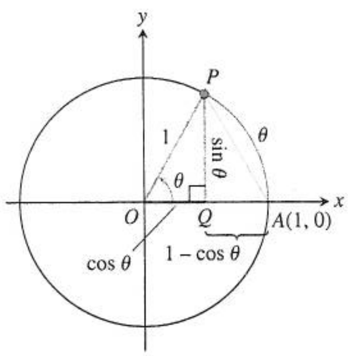



FIGURE 1.49 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 \le \theta^2$.

To establish these inequalities, we picture $\theta$ as a nonzero angle in standard position (Figure 1.49). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc $AP$. The length of line segment $AP$ is therefore less than $|\theta|$.

Triangle $APQ$ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to $\theta^2$:

$$\sin^2\theta \le \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \qquad \text{and} \qquad |1 - \cos\theta| \le |\theta|,$$

so

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

These inequalities will be useful in the next chapter.

## Exercises 1.3

### Radians and Degrees

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) 110°?
2. A central angle in a circle of radius 8 is subtended by an arc of length $10\pi$. Find the angle's radian and degree measures.
3. You want to make an 80° angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

### Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

| $\theta$ | $-\pi$ | $-2\pi/3$ | $0$ | $\pi/2$ | $3\pi/4$ |
|---|---|---|---|---|---|
| $\sin\theta$ | | | | | |
| $\cos\theta$ | | | | | |
| $\tan\theta$ | | | | | |
| $\cot\theta$ | | | | | |
| $\sec\theta$ | | | | | |
| $\csc\theta$ | | | | | |

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

| $\theta$ | $-3\pi/2$ | $-\pi/3$ | $-\pi/6$ | $\pi/4$ | $5\pi/6$ |
|---|---|---|---|---|---|
| $\sin\theta$ | | | | | |
| $\cos\theta$ | | | | | |
| $\tan\theta$ | | | | | |
| $\cot\theta$ | | | | | |
| $\sec\theta$ | | | | | |
| $\csc\theta$ | | | | | |

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if $x$ lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$
8. $\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$
9. $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$
10. $\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$
11. $\tan x = \frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$
12. $\sin x = -\frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

### Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$
14. $\sin(x/2)$
15. $\cos \pi x$
16. $\cos \frac{\pi x}{2}$
17. $-\sin \frac{\pi x}{3}$
18. $-\cos 2\pi x$
19. $\cos\left(x - \frac{\pi}{2}\right)$
20. $\sin\left(x + \frac{\pi}{6}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$ 22. $\cos\left(x + \frac{2\pi}{3}\right) - 2$

Graph the functions in Exercises 23–26 in the *ts*-plane (*t*-axis horizontal, *s*-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$ 24. $s = -\tan \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$ 26. $s = \csc\left(\frac{t}{2}\right)$

**T** 27. **a.** Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

**b.** Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

**T** 28. Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

### Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ 32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ 34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$ 40. $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$ 42. $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin\frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos\frac{11\pi}{12}$ as $\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$.

45. Evaluate $\cos\frac{\pi}{12}$. 46. Evaluate $\sin\frac{5\pi}{12}$.

### Using the Double-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2\frac{\pi}{8}$ 48. $\cos^2\frac{5\pi}{12}$

49. $\sin^2\frac{\pi}{12}$ 50. $\sin^2\frac{3\pi}{8}$

### Solving Trigonometric Equations

For Exercises 51–54, solve for the angle $\theta$, where $0 \le \theta \le 2\pi$.

51. $\sin^2\theta = \frac{3}{4}$ 52. $\sin^2\theta = \cos^2\theta$

53. $\sin 2\theta - \cos\theta = 0$ 54. $\cos 2\theta + \cos\theta = 0$

### Theory and Examples

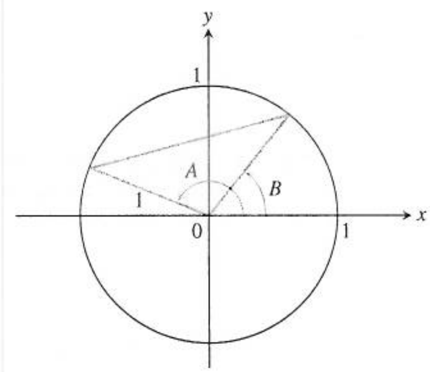
55. **The tangent sum formula** The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

56. (*Continuation of Exercise 55.*) Derive a formula for $\tan(A - B)$.

57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.




58. **a.** Apply the formula for $\cos(A - B)$ to the identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.

**b.** Derive the formula for $\cos(A + B)$ by substituting $-B$ for $B$ in the formula for $\cos(A - B)$ from Exercise 35.

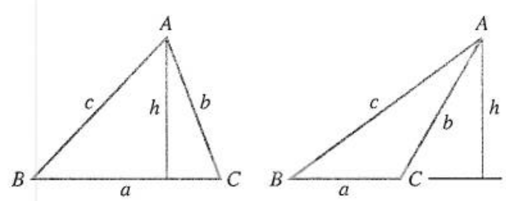
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side $c$.

60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side $c$.

61. **The law of sines** *The law of sines* says that if $a$, $b$, and $c$ are the sides opposite the angles $A$, $B$, and $C$ in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.




62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 59). Find the sine of angle $B$ using the law of sines.

**63.** A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length $a$ of the side opposite $A$.

T **64. The approximation $\sin x \approx x$** It is often useful to know that, when $x$ is measured in radians, $\sin x \approx x$ for numerically small values of $x$. In Section 3.9, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

**a.** With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as $x$ nears the origin?

**b.** With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

### General Sine Curves

For

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify $A$, $B$, $C$, and $D$ for the sine functions in Exercises 65–68 and sketch their graphs.

**65.** $y = 2\sin(x + \pi) - 1$

**66.** $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

**67.** $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

**68.** $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$

### COMPUTER EXPLORATIONS

In Exercises 69–72, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants $A$, $B$, $C$, and $D$. Use a CAS or computer grapher to perform the steps in the exercises.

**69. The period $B$** Set the constants $A = 3$, $C = D = 0$.

**a.** Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

**b.** What happens to the graph for negative values of $B$? Try it with $B = -3$ and $B = -2\pi$.

**70. The horizontal shift $C$** Set the constants $A = 3$, $B = 6$, $D = 0$.

**a.** Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as $C$ increases through positive values.

**b.** What happens to the graph for negative values of $C$?

**c.** What smallest positive value should be assigned to $C$ so the graph exhibits no horizontal shift? Confirm your answer with a plot.

**71. The vertical shift $D$** Set the constants $A = 3$, $B = 6$, $C = 0$.

**a.** Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as $D$ increases through positive values.

**b.** What happens to the graph for negative values of $D$?

**72. The amplitude $A$** Set the constants $B = 6$, $C = D = 0$.

**a.** Describe what happens to the graph of the general sine function as $A$ increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9.

**b.** What happens to the graph for negative values of $A$?

## 1.4 Graphing with Calculators and Computers

A graphing calculator or a computer with graphing software enables us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, care must be taken when using such devices for graphing purposes, and in this section we address some of the issues involved. In Chapter 4 we will see how calculus helps us determine that we are accurately viewing all the important features of a function's graph.

### Graphing Windows

When using a graphing calculator or computer as a graphing tool, a portion of the graph is displayed in a rectangular **display** or **viewing window**. Often the default window gives an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales on both axes are the same. This term does not mean that the display window itself is square (usually it is rectangular), but instead it means that the $x$-unit is the same as the $y$-unit.

When a graph is displayed in the default window, the $x$-unit may differ from the $y$-unit of scaling in order to fit the graph in the window. The viewing window is set by specifying an interval $[a, b]$ for the $x$-values and an interval $[c, d]$ for the $y$-values. The machine selects equally spaced $x$-values in $[a, b]$ and then plots the points $(x, f(x))$. A point is plotted if and only if $x$ lies in the domain of the function and $f(x)$ lies within the interval $[c, d]$. A short line segment is then drawn between each plotted point and its next neighboring point. We now give illustrative examples of some common problems that may occur with this procedure.

**EXAMPLE 1** Graph the function $f(x) = x^3 - 7x^2 + 28$ in each of the following display or viewing windows:

**(a)** $[-10, 10]$ by $[-10, 10]$ **(b)** $[-4, 4]$ by $[-50, 10]$ **(c)** $[-4, 10]$ by $[-60, 60]$

**Solution**

**(a)** We select $a = -10$, $b = 10$, $c = -10$, and $d = 10$ to specify the interval of $x$-values and the range of $y$-values for the window. The resulting graph is shown in Figure 1.50a. It appears that the window is cutting off the bottom part of the graph and that the interval of $x$-values is too large. Let's try the next window.

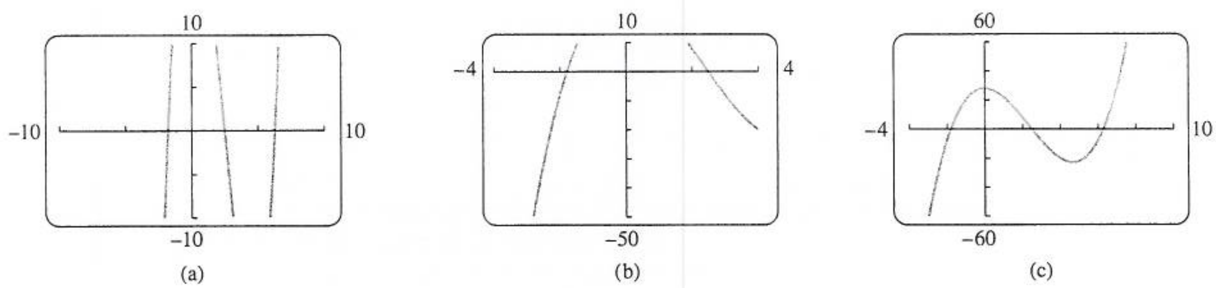



FIGURE 1.50 The graph of $f(x) = x^3 - 7x^2 + 28$ in different viewing windows. Selecting a window that gives a clear picture of a graph is often a trial-and-error process (Example 1).

**(b)** Now we see more features of the graph (Figure 1.50b), but the top is missing and we need to view more to the right of $x = 4$ as well. The next window should help.

**(c)** Figure 1.50c shows the graph in this new viewing window. Observe that we get a more complete picture of the graph in this window, and it is a reasonable graph of a third-degree polynomial. ■

**EXAMPLE 2** When a graph is displayed, the $x$-unit may differ from the $y$-unit, as in the graphs shown in Figures 1.50b and 1.50c. The result is distortion in the picture, which may be misleading. The display window can be made square by compressing or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

Figure 1.51a shows the graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, together with the semicircle $y = \sqrt{9 - x^2}$, in a nonsquare $[-6, 6]$ by $[-6, 8]$ display window. Notice the distortion. The lines do not appear to be perpendicular, and the semicircle appears to be elliptical in shape.

Figure 1.51b shows the graphs of the same functions in a square window in which the $x$-units are scaled to be the same as the $y$-units. Notice that the $[-6, 6]$ by $[-4, 4]$ viewing window has the same $x$-axis in both Figures 1.51a and 1.51b, but the scaling on the $x$-axis has been compressed in Figure 1.51b to make the window square. Figure 1.51c gives an enlarged view of Figure 1.51b with a square $[-3, 3]$ by $[0, 4]$ window. ■

If the denominator of a rational function is zero at some $x$-value within the viewing window, a calculator or graphing computer software may produce a steep near-vertical line segment from the top to the bottom of the window. Here is an example.

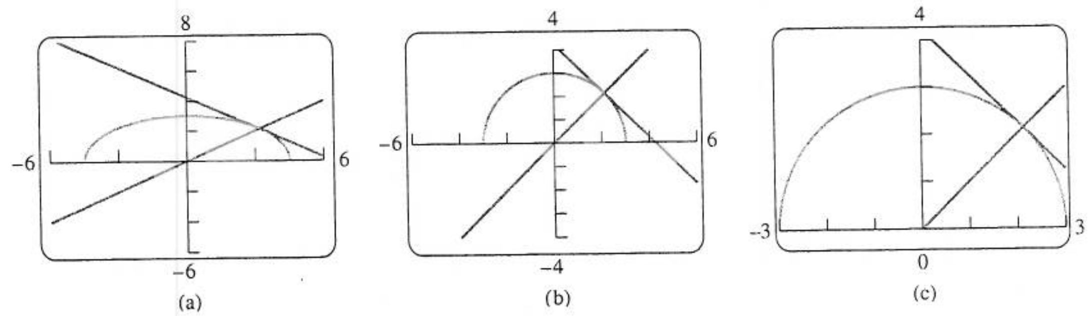



FIGURE 1.51 Graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, and the semicircle $y = \sqrt{9 - x^2}$ appear distorted (a) in a nonsquare window, but clear (b) and (c) in square windows (Example 2).

**EXAMPLE 3** Graph the function $y = \dfrac{1}{2 - x}$.

**Solution** Figure 1.52a shows the graph in the $[-10, 10]$ by $[-10, 10]$ default square window with our computer graphing software. Notice the near-vertical line segment at $x = 2$. It is not truly a part of the graph and $x = 2$ does not belong to the domain of the function. By trial and error we can eliminate the line by changing the viewing window to the smaller $[-6, 6]$ by $[-4, 4]$ view, revealing a better graph (Figure 1.52b). ■

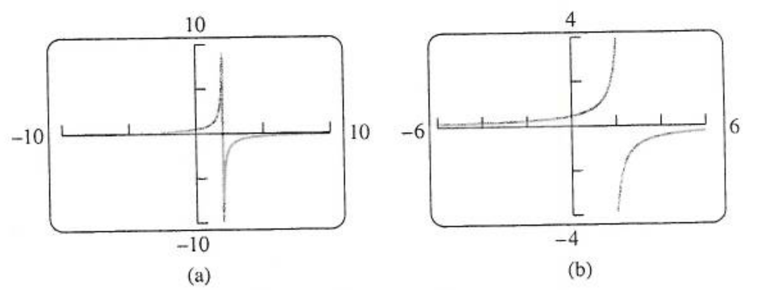



FIGURE 1.52 Graphs of the function $y = \dfrac{1}{2 - x}$. A vertical line may appear without a careful choice of the viewing window (Example 3).

Sometimes the graph of a trigonometric function oscillates very rapidly. When a calculator or computer software plots the points of the graph and connects them, many of the maximum and minimum points are actually missed. The resulting graph is then very misleading.

**EXAMPLE 4** Graph the function $f(x) = \sin 100x$.

**Solution** Figure 1.53a shows the graph of $f$ in the viewing window $[-12, 12]$ by $[-1, 1]$. We see that the graph looks very strange because the sine curve should oscillate periodically between $-1$ and 1. This behavior is not exhibited in Figure 1.53a. We might

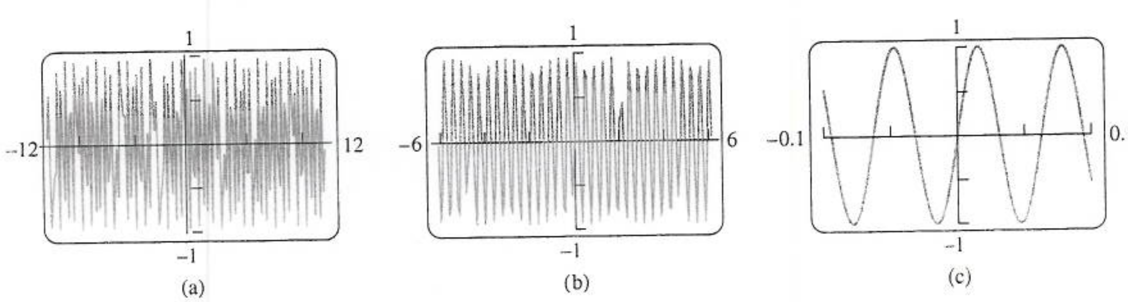



FIGURE 1.53 Graphs of the function $y = \sin 100x$ in three viewing windows. Because the period is $2\pi/100 \approx 0.063$, the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 4).

experiment with a smaller viewing window, say $[-6, 6]$ by $[-1, 1]$, but the graph is not better (Figure 1.53b). The difficulty is that the period of the trigonometric function $y = \sin 100x$ is very small $(2\pi/100 \approx 0.063)$. If we choose the much smaller viewing window $[-0.1, 0.1]$ by $[-1, 1]$ we get the graph shown in Figure 1.53c. This graph reveals the expected oscillations of a sine curve. ■

**EXAMPLE 5** Graph the function $y = \cos x + \frac{1}{50}\sin 50x$.

**Solution** In the viewing window $[-6, 6]$ by $[-1, 1]$ the graph appears much like the cosine function with some small sharp wiggles on it (Figure 1.54a). We get a better look when we significantly reduce the window to $[-0.6, 0.6]$ by $[0.8, 1.02]$, obtaining the graph in Figure 1.54b. We now see the small but rapid oscillations of the second term, $1/50 \sin 50x$, added to the comparatively larger values of the cosine curve. ■

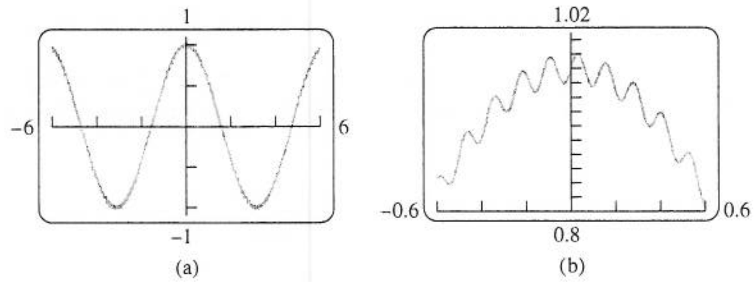



FIGURE 1.54 In (b) we see a close-up view of the function $y = \cos x + \frac{1}{50}\sin 50x$ graphed in (a). The term $\cos x$ clearly dominates the second term, $\frac{1}{50}\sin 50x$, which produces the rapid oscillations along the cosine curve. Both views are needed for a clear idea of the graph (Example 5).

## Obtaining a Complete Graph

Some graphing devices will not display the portion of a graph for $f(x)$ when $x < 0$. Usually that happens because of the procedure the device is using to calculate the function values. Sometimes we can obtain the complete graph by defining the formula for the function in a different way.

**EXAMPLE 6** Graph the function $y = x^{1/3}$.

**Solution** Some graphing devices display the graph shown in Figure 1.55a. When we compare it with the graph of $y = x^{1/3} = \sqrt[3]{x}$ in Figure 1.17, we see that the left branch for

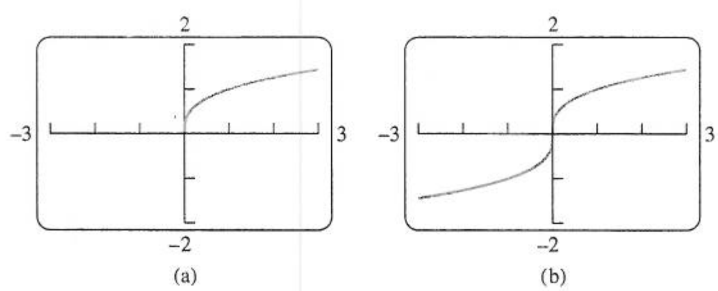



FIGURE 1.55 The graph of $y = x^{1/3}$ is missing the left branch in (a). In (b) we graph the function $f(x) = \frac{x}{|x|}\cdot|x|^{1/3}$, obtaining both branches. (See Example 6.)

$x < 0$ is missing. The reason the graphs differ is that many calculators and computer software programs calculate $x^{1/3}$ as $e^{(1/3)\ln x}$. Since the logarithmic function is not defined for negative values of $x$, the computing device can produce only the right branch, where $x > 0$. (Logarithmic and exponential functions are presented in Chapter 7.)

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals $x^{1/3}$ except at $x = 0$ (where $f$ is undefined, although $0^{1/3} = 0$). The graph of $f$ is shown in Figure 1.55b. ■

## Exercises 1.4

### Choosing a Viewing Window

T In Exercises 1–4, use a graphing calculator or computer to determine which of the given viewing windows displays the most appropriate graph of the specified function.

**1.** $f(x) = x^4 - 7x^2 + 6x$

**a.** $[-1, 1]$ by $[-1, 1]$ **b.** $[-2, 2]$ by $[-5, 5]$

**c.** $[-10, 10]$ by $[-10, 10]$ **d.** $[-5, 5]$ by $[-25, 15]$

**2.** $f(x) = x^3 - 4x^2 - 4x + 16$

**a.** $[-1, 1]$ by $[-5, 5]$ **b.** $[-3, 3]$ by $[-10, 10]$

**c.** $[-5, 5]$ by $[-10, 20]$ **d.** $[-20, 20]$ by $[-100, 100]$

**3.** $f(x) = 5 + 12x - x^3$

**a.** $[-1, 1]$ by $[-1, 1]$ **b.** $[-5, 5]$ by $[-10, 10]$

**c.** $[-4, 4]$ by $[-20, 20]$ **d.** $[-4, 5]$ by $[-15, 25]$

**4.** $f(x) = \sqrt{5 + 4x - x^2}$

**a.** $[-2, 2]$ by $[-2, 2]$ **b.** $[-2, 6]$ by $[-1, 4]$

**c.** $[-3, 7]$ by $[0, 10]$ **d.** $[-10, 10]$ by $[-10, 10]$

### Finding a Viewing Window

T In Exercises 5–30, find an appropriate viewing window for the given function and use it to display its graph.

**5.** $f(x) = x^4 - 4x^3 + 15$ **6.** $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$

**7.** $f(x) = x^5 - 5x^4 + 10$ **8.** $f(x) = 4x^3 - x^4$

**9.** $f(x) = x\sqrt{9 - x^2}$ **10.** $f(x) = x^2(6 - x^3)$

**11.** $y = 2x - 3x^{2/3}$ **12.** $y = x^{1/3}(x^2 - 8)$

**13.** $y = 5x^{2/5} - 2x$ **14.** $y = x^{2/3}(5 - x)$

**15.** $y = |x^2 - 1|$ **16.** $y = |x^2 - x|$

**17.** $y = \frac{x + 3}{x + 2}$ **18.** $y = 1 - \frac{1}{x + 3}$

**19.** $f(x) = \frac{x^2 + 2}{x^2 + 1}$ **20.** $f(x) = \frac{x^2 - 1}{x^2 + 1}$

**21.** $f(x) = \frac{x - 1}{x^2 - x - 6}$ **22.** $f(x) = \frac{8}{x^2 - 9}$

**23.** $f(x) = \frac{6x^2 - 15x + 6}{4x^2 - 10x}$ **24.** $f(x) = \frac{x^2 - 3}{x - 2}$

**25.** $y = \sin 250x$ **26.** $y = 3\cos 60x$

**27.** $y = \cos\left(\frac{x}{50}\right)$ **28.** $y = \frac{1}{10}\sin\left(\frac{x}{10}\right)$

**29.** $y = x + \frac{1}{10}\sin 30x$ **30.** $y = x^2 + \frac{1}{50}\cos 100x$

**31.** Graph the lower half of the circle defined by the equation $x^2 + 2x = 4 + 4y - y^2$.

**32.** Graph the upper branch of the hyperbola $y^2 - 16x^2 = 1$.

**33.** Graph four periods of the function $f(x) = -\tan 2x$.

**34.** Graph two periods of the function $f(x) = 3\cot\frac{x}{2} + 1$.

**35.** Graph the function $f(x) = \sin 2x + \cos 3x$.

**36.** Graph the function $f(x) = \sin^3 x$.

### Graphing in Dot Mode

T Another way to avoid incorrect connections when using a graphing device is through the use of a "dot mode," which plots only the points. If your graphing utility allows that mode, use it to plot the functions in Exercises 37–40.

**37.** $y = \frac{1}{x - 3}$ **38.** $y = \sin\frac{1}{x}$

**39.** $y = x\lfloor x \rfloor$ **40.** $y = \frac{x^3 - 1}{x^2 - 1}$

## Chapter 1 Questions to Guide Your Review

1. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
2. What is the graph of a real-valued function of a real variable? What is the vertical line test?
3. What is a piecewise-defined function? Give examples.
4. What are the important types of functions frequently encountered in calculus? Give an example of each type.

5. What is meant by an increasing function? A decreasing function? Give an example of each.
6. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
7. If $f$ and $g$ are real-valued functions, how are the domains of $f + g, f - g, fg$, and $f/g$ related to the domains of $f$ and $g$? Give examples.
8. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
9. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
10. How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
11. What is the standard equation of an ellipse with center $(h, k)$? What is its major axis? Its minor axis? Give examples.
12. What is radian measure? How do you convert from radians to degrees? Degrees to radians?
13. Graph the six basic trigonometric functions. What symmetries do the graphs have?
14. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
15. Starting with the identity $\sin^2\theta + \cos^2\theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
16. How does the formula for the general sine function $f(x) = A\sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants $A$, $B$, $C$, and $D$.
17. Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.

# Chapter 1 Practice Exercises

## Functions and Graphs

1. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
2. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
3. A point $P$ in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of $P$ as functions of the angle of inclination of the line joining $P$ to the origin.
4. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the $y$-axis, the origin, or neither.

5. $y = x^{1/5}$
6. $y = x^{2/5}$
7. $y = x^2 - 2x - 1$
8. $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

9. $y = x^2 + 1$
10. $y = x^5 - x^3 - x$
11. $y = 1 - \cos x$
12. $y = \sec x \tan x$
13. $y = \dfrac{x^4 + 1}{x^3 - 2x}$
14. $y = x - \sin x$
15. $y = x + \cos x$
16. $y = x\cos x$

17. Suppose that $f$ and $g$ are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?

**a.** $fg$ **b.** $f^3$ **c.** $f(\sin x)$ **d.** $g(\sec x)$ **e.** $|g|$

18. If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–28, find the **(a)** domain and **(b)** range.

19. $y = |x| - 2$
20. $y = -2 + \sqrt{1 - x}$
21. $y = \sqrt{16 - x^2}$
22. $y = 3^{2-x} + 1$
23. $y = 2e^{-x} - 3$
24. $y = \tan(2x - \pi)$
25. $y = 2\sin(3x + \pi) - 1$
26. $y = x^{2/5}$
27. $y = \ln(x - 3) + 1$
28. $y = -1 + \sqrt[3]{2 - x}$

29. State whether each function is increasing, decreasing, or neither.
    - **a.** Volume of a sphere as a function of its radius
    - **b.** The greatest integer function
    - **c.** Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)
    - **d.** Kinetic energy as a function of a particle's velocity

30. Find the largest interval on which the given function is increasing.
    - **a.** $f(x) = |x - 2| + 1$
    - **b.** $f(x) = (x + 1)^4$
    - **c.** $g(x) = (3x - 1)^{1/3}$
    - **d.** $R(x) = \sqrt{2x - 1}$

## Piecewise-Defined Functions

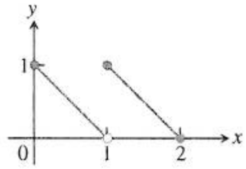
In Exercises 31 and 32, find the **(a)** domain and **(b)** range.

31. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

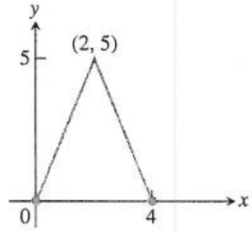
32. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 33 and 34, write a piecewise formula for the function.

**33.**




**34.**




### Composition of Functions

In Exercises 35 and 36, find

**a.** $(f \circ g)(-1)$. **b.** $(g \circ f)(2)$.
**c.** $(f \circ f)(x)$. **d.** $(g \circ g)(x)$.

**35.** $f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{\sqrt{x+2}}$

**36.** $f(x) = 2 - x, \quad g(x) = \sqrt[3]{x+1}$

In Exercises 37 and 38, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

**37.** $f(x) = 2 - x^2, \quad g(x) = \sqrt{x+2}$

**38.** $f(x) = \sqrt{x}, \quad g(x) = \sqrt{1-x}$

For Exercises 39 and 40, sketch the graphs of $f$ and $f \circ f$.

**39.** $f(x) = \begin{cases} -x-2, & -4 \le x \le -1 \\ -1, & -1 < x \le 1 \\ x-2, & 1 < x \le 2 \end{cases}$

**40.** $f(x) = \begin{cases} x+1, & -2 \le x < 0 \\ x-1, & 0 \le x \le 2 \end{cases}$

**Composition with absolute values** In Exercises 41–48, graph $f_1$ and $f_2$ together. Then describe how applying the absolute value function in $f_2$ affects the graph of $f_1$.

| | $f_1(x)$ | $f_2(x)$ |
|---|---|---|
| **41.** | $x$ | $\lvert x\rvert$ |
| **42.** | $x^2$ | $\lvert x\rvert^2$ |
| **43.** | $x^3$ | $\lvert x^3\rvert$ |
| **44.** | $x^2 + x$ | $\lvert x^2 + x\rvert$ |
| **45.** | $4 - x^2$ | $\lvert 4 - x^2\rvert$ |
| **46.** | $\frac{1}{x}$ | $\frac{1}{\lvert x\rvert}$ |
| **47.** | $\sqrt{x}$ | $\sqrt{\lvert x\rvert}$ |
| **48.** | $\sin x$ | $\sin\lvert x\rvert$ |

### Shifting and Scaling Graphs

**49.** Suppose the graph of $g$ is given. Write equations for the graphs that are obtained from the graph of $g$ by shifting, scaling, or reflecting, as indicated.

**a.** Up $\frac{1}{2}$ unit, right 3

**b.** Down 2 units, left $\frac{2}{3}$

**c.** Reflect about the $y$-axis

**d.** Reflect about the $x$-axis

**e.** Stretch vertically by a factor of 5

**f.** Compress horizontally by a factor of 5

**50.** Describe how each graph is obtained from the graph of $y = f(x)$.

**a.** $y = f(x-5)$ **b.** $y = f(4x)$
**c.** $y = f(-3x)$ **d.** $y = f(2x+1)$
**e.** $y = f\left(\frac{x}{3}\right) - 4$ **f.** $y = -3f(x) + \frac{1}{4}$

In Exercises 51–54, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

**51.** $y = -\sqrt{1 + \frac{x}{2}}$ **52.** $y = 1 - \frac{x}{3}$

**53.** $y = \frac{1}{2x^2} + 1$ **54.** $y = (-5x)^{1/3}$

### Trigonometry

In Exercises 55–58, sketch the graph of the given function. What is the period of the function?

**55.** $y = \cos 2x$ **56.** $y = \sin\frac{x}{2}$

**57.** $y = \sin \pi x$ **58.** $y = \cos\frac{\pi x}{2}$

**59.** Sketch the graph $y = 2\cos\left(x - \frac{\pi}{3}\right)$.

**60.** Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 61–64, $ABC$ is a right triangle with the right angle at $C$. The sides opposite angles $A$, $B$, and $C$ are $a$, $b$, and $c$, respectively.

**61. a.** Find $a$ and $b$ if $c = 2, B = \pi/3$.
**b.** Find $a$ and $c$ if $b = 2, B = \pi/3$.

**62. a.** Express $a$ in terms of $A$ and $c$.
**b.** Express $a$ in terms of $A$ and $b$.

**63. a.** Express $a$ in terms of $B$ and $b$.
**b.** Express $c$ in terms of $A$ and $a$.

**64. a.** Express $\sin A$ in terms of $a$ and $c$.
**b.** Express $\sin A$ in terms of $b$ and $c$.

**65. Height of a pole** Two wires stretch from the top $T$ of a vertical pole to points $B$ and $C$ on the ground, where $C$ is 10 m closer to the base of the pole than is $B$. If wire $BT$ makes an angle of 35° with the horizontal and wire $CT$ makes an angle of 50° with the horizontal, how high is the pole?

**66. Height of a weather balloon** Observers at positions $A$ and $B$ 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between $A$ and $B$, find the height of the balloon.

T **67. a.** Graph the function $f(x) = \sin x + \cos(x/2)$.
**b.** What appears to be the period of this function?
**c.** Confirm your finding in part (b) algebraically.

T **68. a.** Graph $f(x) = \sin(1/x)$.
**b.** What are the domain and range of $f$?
**c.** Is $f$ periodic? Give reasons for your answer.

# Chapter 1 Additional and Advanced Exercises

## Functions and Graphs

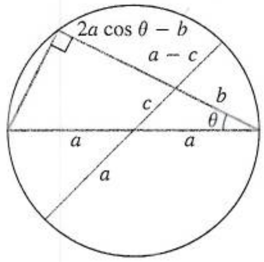
1. Are there two functions $f$ and $g$ such that $f \circ g = g \circ f$? Give reasons for your answer.
2. Are there two functions $f$ and $g$ with the following property? The graphs of $f$ and $g$ are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.
3. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if $f$ is even instead? Give reasons for your answer.
4. If $g(x)$ is an odd function defined for all values of $x$, can anything be said about $g(0)$? Give reasons for your answer.
5. Graph the equation $|x| + |y| = 1 + x$.
6. Graph the equation $y + |y| = x + |x|$.

## Derivations and Proofs

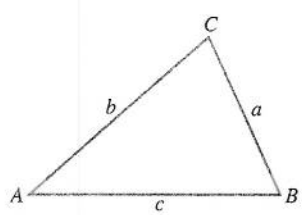
7. Prove the following identities.

   a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$ b. $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

8. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)




9. Show that the area of triangle $ABC$ is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.




10. Show that the area of triangle $ABC$ is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.
11. Show that if $f$ is both even and odd, then $f(x) = 0$ for every $x$ in the domain of $f$.
12. a. **Even-odd decompositions** Let $f$ be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever $x$ does. Show that $f$ is the sum of an even function and an odd function:

    $$f(x) = E(x) + O(x),$$

    where $E$ is an even function and $O$ is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that $E$ is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

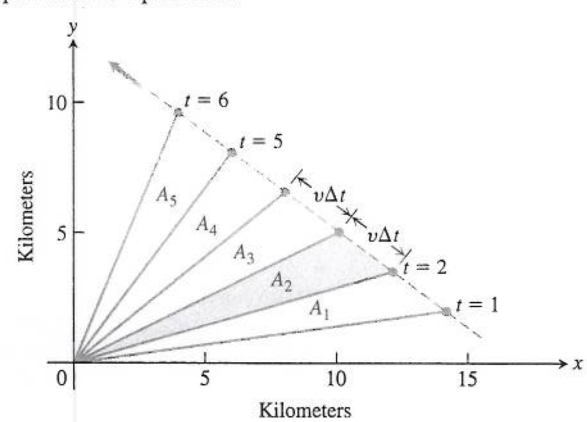
    b. **Uniqueness** Show that there is only one way to write $f$ as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where $E_1$ is even and $O_1$ is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

## Grapher Explorations—Effects of Parameters

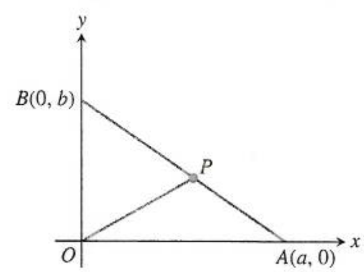
13. What happens to the graph of $y = ax^2 + bx + c$ as
    a. $a$ changes while $b$ and $c$ remain fixed?
    b. $b$ changes ($a$ and $c$ fixed, $a \neq 0$)?
    c. $c$ changes ($a$ and $b$ fixed, $a \neq 0$)?
14. What happens to the graph of $y = a(x + b)^3 + c$ as
    a. $a$ changes while $b$ and $c$ remain fixed?
    b. $b$ changes ($a$ and $c$ fixed, $a \neq 0$)?
    c. $c$ changes ($a$ and $b$ fixed, $a \neq 0$)?

## Geometry

15. An object's center of mass moves at a constant velocity $v$ along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



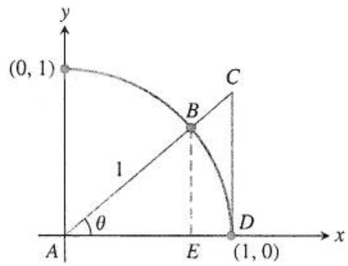

16. a. Find the slope of the line from the origin to the midpoint $P$ of side $AB$ in the triangle in the accompanying figure $(a, b > 0)$.




    b. When is $OP$ perpendicular to $AB$?

17. Consider the quarter-circle of radius 1 and right triangles $ABE$ and $ACD$ given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2}\sin\theta\cos\theta < \frac{\theta}{2} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}.$$




18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants $a$, $b$, $c$, $d$ in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of $x$?

## Chapter 1 Technology Application Projects

***An Overview of Mathematica***
An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.

Mathematica/Maple Module:

***Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals***
Construct and interpret mathematical models, analyze and improve them, and make predictions using them.