Section 1.3

Mathematics as Problem Solving

The Role of Problem Solving ▪ A Problem-Solving Model ▪ Problem-Solving Strategies

In this section, we explore the notion of mathematics as problem solving. We also look at the meaning of a problem, a model for the problem-solving process, and some useful problem-solving strategies.

Essential Understandings for Section 1.3

- One of the most important processes in doing mathematics is problem solving.
- Although there are many ways to solve problems, it is helpful to have in mind a model of how solving problems works.
- There are specific strategies that can be very helpful in solving problems.

The Role of Problem Solving

When people think of mathematics as solving problems, they view mathematics primarily as a thinking process and computation and other rule-oriented procedures as tools to be used when solving problems. Problem solving is central to the development and application of mathematics and is used extensively in all branches of mathematics. When someone is "doing mathematics," they are involved in the process of solving problems. Let's explore further the role of problem solving in mathematics and how to communicate with others effectively about problem solving. First, we need to look at the meaning of a problem and the problem-solving process.

The Meaning of a Problem. Although many possible answers exist to the question, "What is a problem?" the following description will serve to focus your attention on some key ideas:

Description of a Problem

A **problem** is a situation for which the following conditions exist:

a. It involves a question that represents a challenge for the individual.
b. The question cannot be answered immediately by some routine procedures known to the individual.
c. The individual accepts the challenge.

One key idea implied by this description is that what is a problem for one person may not be a problem for another person. For example, because of (a), (b), or (c) in the description, finding the product 10×12 is not a problem for you, but it probably would be a problem for the average second-grader.

Another key idea is that the word *situation* in the description may be interpreted very broadly and that many different kinds of problems exist. Clearly, a problem can be much more than just a standard word problem. For example, the following questions suggest some real-world situations that can present a challenge and are definitely problems:

> How can I fix my car so it will run?
>
> What is the most efficient way to travel by car to six different cities and return home?
>
> What will be the estimated total cost for my junior year in college?

Other questions, such as "Can every map be colored with only four different colors if regions that have a border in common must be colored differently?" have been challenging mathematicians for centuries. This particular problem was solved recently, but several extremely important problems in mathematics remain unsolved. Even puzzle questions, such as "How can you cut a cake into eight pieces with three straight cuts?" legitimately qualify as problems. These examples, and others that you may generate, reinforce the need to take a broad view of the meaning of a problem. Mini-Investigation 1.7 encourages you to think about what constitutes a problem.

Write a paragraph describing your reaction and your solution, if any, to the problem.

MINI-INVESTIGATION 1.7 Solving a Problem

If you use the criteria for a problem presented in the description box, why would the following question be or not be a problem for you?

> If the letters *a*–*z* are associated with the monetary values 1¢–26¢, what $1 words, if any, can you find?

The Meaning of Problem Solving. With the meaning of a problem in mind, let's now consider the meaning of problem solving.

Description of Problem Solving

Problem solving is a *process* by which an individual uses previously learned concepts, facts, and relationships, along with various reasoning skills and strategies, to answer a question or questions about a situation.

Thinking of problem solving as a process requires distinguishing it from a computational process, such as adding two 3-digit numbers. The process of adding numbers can be made into an *algorithm*, in which a sequence of prescribed steps always can be followed in a certain order to produce an answer. However, the problem-solving process cannot be made into an algorithm. Because of the many different varieties of problems, no single sequence of steps can always be used to solve a problem. Even though some helpful general guidelines and strategies are available, problem solving is a nonalgorithmic process that requires creativity, thought, and judgment.

Along with guidelines and strategies for exploring problems, many tools support the problem-solving process. Pencil and paper, compasses and protractors, rulers, random-number tables, calculators, geometry exploration software, and other computer programs are but a few of the many tools available. As you solve problems, you need to be aware of the situations in which these tools can help you achieve the desired results.

Note that, when discussing the problem-solving process, we distinguish between an **answer** to a problem, which is the final result, and a **solution** to a problem, which is the process used to find the answer. Thus, when asked to give your answer to a problem, you might give only the number 25. But when asked to give your solution to the problem, you might describe how you drew a picture, made a table, and extended a pattern to obtain the number 25.

When solving problems, you shouldn't look for a standard solution that others have developed. Rather, you should generate a solution based on your unique way of thinking. Example 1.12 illustrates that even a standard word problem can be solved in a variety of ways, each of which is appropriate.

Example 1.12 Problem Solving: Slices of Pizza

Gene, Tina, and Brenda have ordered a large pizza. Gene will eat three times as much pizza as Tina. Tina will eat twice as much pizza as Brenda. If the pizza is cut into 36 slices, how many slices will each person get?

SOLUTION

Dede's thinking: When Gene gets 6 slices, Tina gets 2, and Brenda gets 1. But that totals only 9 slices, not 36. I want four times this number of slices so I use 24 slices for Gene instead of 6. Then Tina gets 8, and Brenda gets 4. I now have a total of 36 slices, which is the answer to the problem.

Megan's thinking: I fill in the columns of a table for Brenda and Tina first because Tina always gets twice as many slices as Brenda. Then I fill in the column for Gene with the remainder of the 36 slices. I just add rows to the table until Gene has three times as many slices as Tina.

Brenda	Tina	Gene
1	2	33
2	4	30
3	6	27

The numbers in the table are correct when Brenda gets 4 slices, Tina gets 8 slices, and Gene gets 24 slices.

Judd's thinking: I used what I learned in elementary algebra and wrote an equation to solve the problem. I let s equal the number of slices that Brenda ate, $2s$ equal the number of slices that Tina ate, and $6s$ equal the number of slices that Gene ate. Then, I solved the equation

$$\begin{aligned} s + 2s + 6s &= 36 \\ 9s &= 36 \\ s &= 4. \end{aligned}$$

So Brenda ate 4 slices, Tina ate 8 slices, and Gene ate 24 slices.

YOUR TURN

Practice: Find at least two different solutions to the following problem:

How many people can be seated at a large table formed by lining up eight square tables, each of which ordinarily seats four people, end to end?

Reflect: Which of the three solutions to the example problem do you think is "best?" Do you believe everyone who answers this question would agree with you? Why? ■

Exercise 30 on p. 51 provides an opportunity for you to use technology, if available, in the problem-solving process.

A Problem-Solving Model

Although no set rules exist for solving a problem, a general approach to this task has been developed. The following problem-solving model has four main phases. They are similar to those described by George Pólya (1945) in his classic book on problem solving, *How to Solve It*. Each of these phases may involve the use of some important problem-solving skills and problem-solving strategies, which are also shown in the model in Figure 1.6.

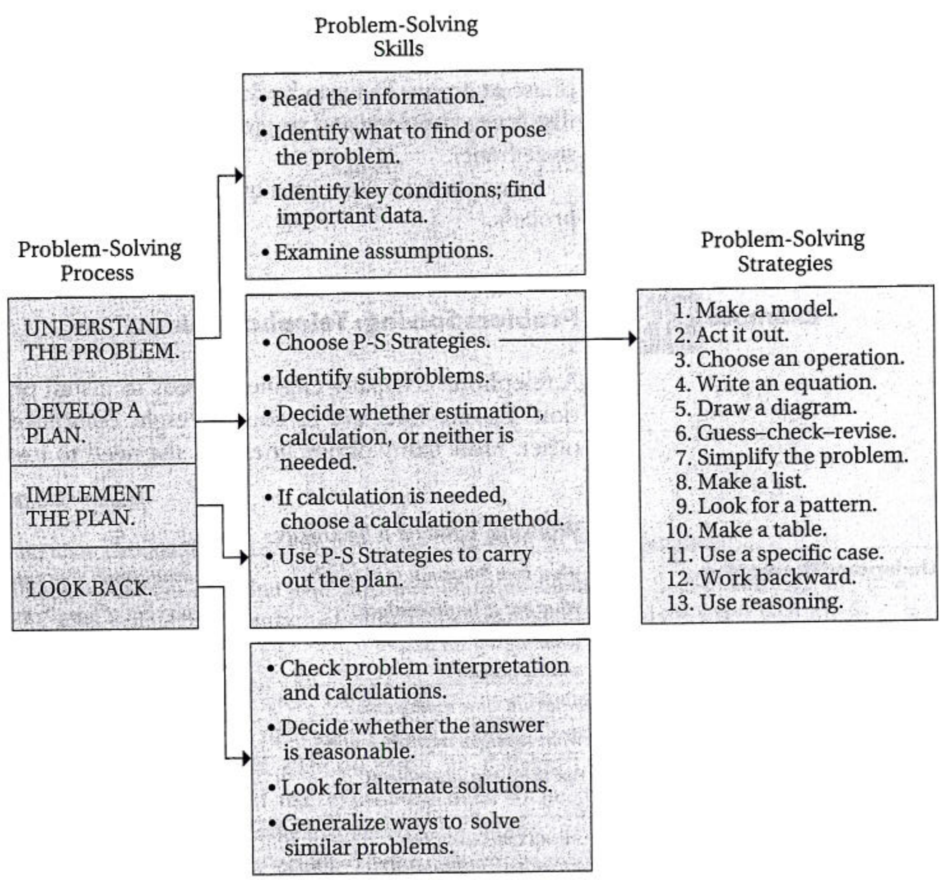


FIGURE 1.6
Problem-solving model.

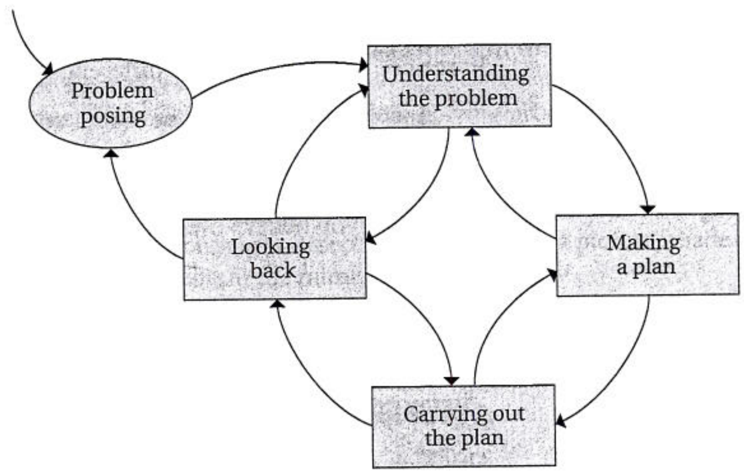


FIGURE 1.7
The flexibility inherent in the problem-solving process.

The process described in the model certainly is not a lock-step process, and flexibility across phases is important. The model contains sequential steps, but when actually solving a problem, you could legitimately deal with more than one phase at a time or jump back and forth among phases. The diagram in Figure 1.7 illustrates this idea and suggests the importance of correctly formulating problem statements.

Example 1.13 illustrates use of the model in Figure 1.7 in the problem-solving process.

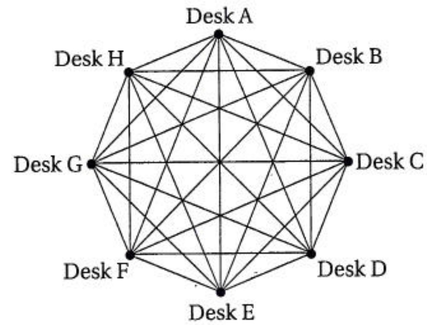
Example 1.13 Problem Solving: Telephone Lines

A telephone company engineer needs to install phone lines for a political convention. Private lines are to connect eight committee chairpersons' desks with each other. How many phone lines will she need to install?

Working Toward a Solution

Understand the problem	*What does the situation involve?*	Installing phone lines between desks.
	What has to be determined?	The number of lines to be installed.
	What are the key data and conditions?	Eight committee chairpersons' desks are to be connected by private lines.
	What are some assumptions?	A separate line is needed for each two-desk connection.
Develop a plan	*What strategies might be useful?*	Draw a diagram.
	Are there any subproblems?	Yes. How many lines are needed to connect each desk to all the other desks?
	Should the answer be estimated or calculated?	Calculate the exact numbers.
	What method of calculation should be used?	Use mental math.

Implement the plan	*How should the strategies be used?*	Draw a diagram showing 7 lines coming from each desk.



For 8 desks, $8 \times 7 = 56$. However, the line from desk A to desk B, for example, is the same as the line from desk B to desk A, so the number of lines needed is only $56 \div 2 = 28$.

	What is the answer?	The answer is that 28 telephone lines will be needed.
Look back	*Is the interpretation correct?*	*Check:* The diagram fits the data and conditions.
	Is the calculation correct?	*Check:* The mental math was correct.
	Is the answer reasonable?	Yes. It has to be much less than 56, as this number represents 7 lines from 8 desks without considering repetition.
	Is there another way to solve the problem?	Make a table of values for 2, 3, and 4 desks and look for a pattern to help extend the table.

Desks	**Lines**
2	1
3	3
4	6
5	?
6	?
7	?
8	?

YOUR TURN

Practice: Solve the problem when 10 desks are to be connected.

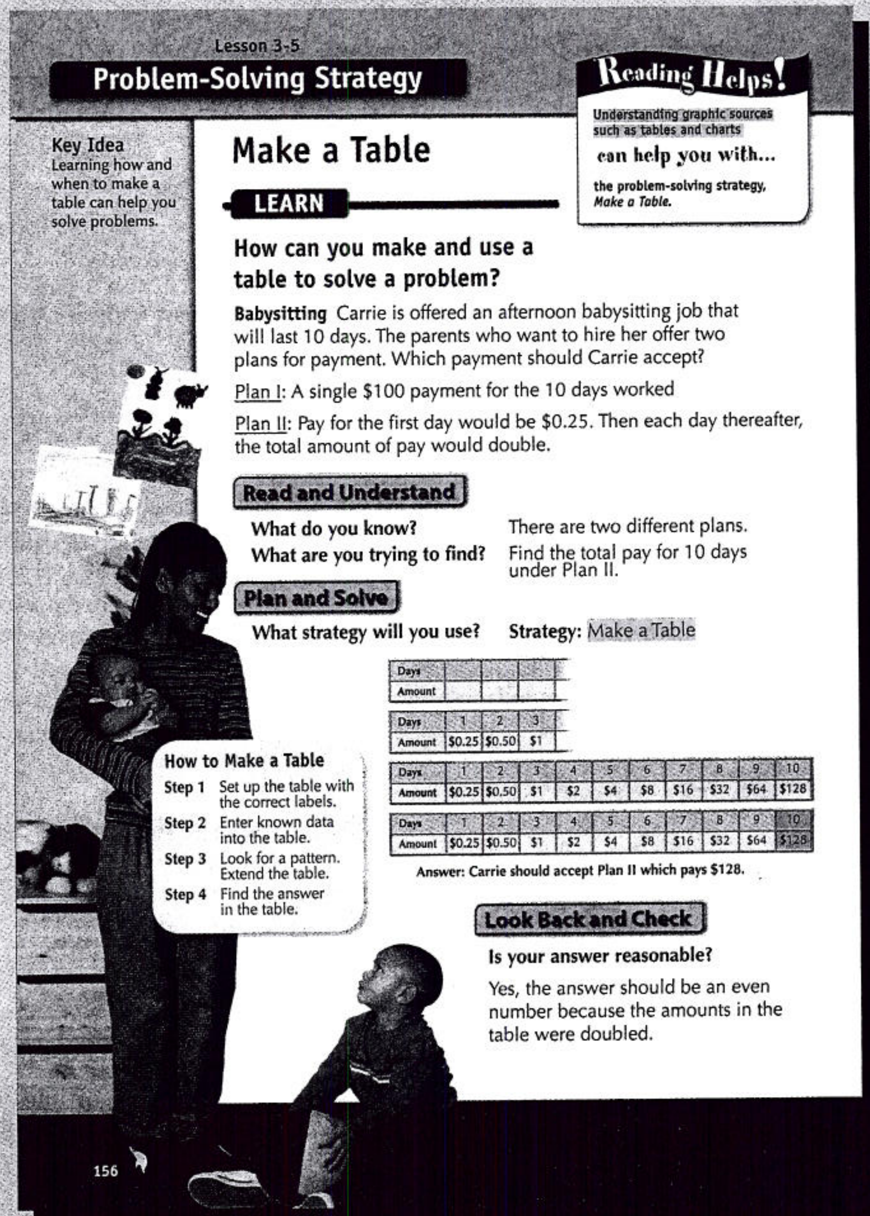
Reflect: If you say that the number of lines needed is $\frac{1}{2}$ the number of desks times 1 less than the number of desks, use examples to explain what you mean. ■

Another important aspect of the problem-solving model is the use of estimation. **Estimation** is the process of determining an answer that is reasonably close to the exact answer. It can be used at different stages in problem solving. For example, you could use estimated data to help you understand a problem. Also, when developing and implementing a plan, you could choose the problem-solving strategy of guess–check–revise, which might involve estimation. You could also decide that an estimated answer is all that you need when finding a solution. Finally, when looking back, you could use estimation to help you decide whether

Connection to the PreK–8 Classroom

Analyzing a Textbook Page: Solving a Problem

The following page from an elementary mathematics textbook focuses on one of the problem-solving strategies given in the problem-solving model (see Figure 1.6), namely, making a table.



Lesson 3-5

Problem-Solving Strategy

Key Idea
Learning how and when to make a table can help you solve problems.

Make a Table

Reading Helps!

Understanding graphic sources such as tables and charts

can help you with...

the problem-solving strategy, *Make a Table.*

LEARN

How can you make and use a table to solve a problem?

Babysitting Carrie is offered an afternoon babysitting job that will last 10 days. The parents who want to hire her offer two plans for payment. Which payment should Carrie accept?

Plan I: A single $100 payment for the 10 days worked

Plan II: Pay for the first day would be $0.25. Then each day thereafter, the total amount of pay would double.

Read and Understand

What do you know? There are two different plans.

What are you trying to find? Find the total pay for 10 days under Plan II.

Plan and Solve

What strategy will you use? **Strategy:** Make a Table

Days			
Amount			

Days	1	2	3
Amount	$0.25	$0.50	$1

Days	1	2	3	4	5	6	7	8	9	10
Amount	$0.25	$0.50	$1	$2	$4	$8	$16	$32	$64	$128

Days	1	2	3	4	5	6	7	8	9	10
Amount	$0.25	$0.50	$1	$2	$4	$8	$16	$32	$64	$128

Answer: Carrie should accept Plan II which pays $128.

How to Make a Table

Step 1 Set up the table with the correct labels.
Step 2 Enter known data into the table.
Step 3 Look for a pattern. Extend the table.
Step 4 Find the answer in the table.

Look Back and Check

Is your answer reasonable?

Yes, the answer should be an even number because the amounts in the table were doubled.

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FIGURE 1.8
Source: *Scott Foresman–Addison Wesley Mathematics, Grade 6,* p. 156. © 2004 Pearson Education. Reprinted with permission.

Do you think this strategy was an effective way to solve the problem? Explain.

the answer is reasonable. The Estimation Application features that appear throughout this book provide an opportunity for you to improve your estimating skills in problem solving.

Estimation Application

The Dripping Faucet
Consider the following problem.

How much extra does it cost for the water lost in 1 year from a dripping faucet?

a. Estimate the data needed for the problem.
b. Assume that the cost of water is 4 gallons for 1¢ and estimate the answer to the question.

Determine how to check your estimate in (b).

Problem-Solving Strategies

When solving the pizza or telephone line problems in Examples 1.12 and 1.13, you may have guessed, checked, and revised your guess. Or you may have drawn a diagram, solved a simpler problem, made a table, solved an equation, or used other approaches. All these approaches are the problem-solving strategies shown in the model in Figure 1.6 and summarized in the following list:

Problem-Solving Strategies

- Make a model.
- Act it out.
- Choose an operation.
- Write an equation.
- Draw a diagram.
- Guess–check–revise.
- Simplify the problem.
- Make a list.
- Look for a pattern.
- Make a table.
- Use a specific case.
- Work backward.
- Use reasoning.

Example 1.14 illustrates the use of some of these strategies and provides practice in the use of others.

Example 1.14 **Problem Solving: The Bicycle-Built-for-Two Rides**

Each of six friends takes a bicycle-built-for-two ride with everyone else in the group. How many bike rides do they take?

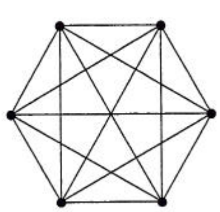
SOLUTION

Bill's thinking: Six of us can pair up and count the possibilities *(act out)*.

Christine's thinking: We can use colored crayons as bike riders and count possible pairs *(make a model)*.

Tom's thinking: Why not write down all possible pairs—Terry, Lynn; Terry, Phyllis; and so on *(make a list)*.

Janie's thinking: First, find out how many rides if there were three friends. Once we figure out how to solve this simpler problem, we can consider the original problem *(solve a simpler problem)*.



John's thinking: Drawing a picture is easier. Dots are friends and lines are rides *(draw a diagram)*.

Ike's thinking: We could show simpler cases in a table *(make a table)* and then extend the table *(look for a pattern)*.

Number of Friends	Number of Rides
1	0
2	1
3	3
4	6

Jerry's thinking: Each of six friends would ride with five other friends, making 6×5 rides. Randy riding with Peggy is the same as Peggy riding with Randy, so there are half that many rides *(use reasoning)*.

YOUR TURN

Practice: Use the strategies listed to solve the following problem:

Donna put half of her babysitting earnings in a savings account. Of the remaining amount, she gave $7 to a Help the Hungry fund, leaving her with $2. How much money did she save originally?

a. *Work backward* and *choose operation(s)*. (*Hint:* Start with $2 and work backward. Decide which operations you need to get back to the beginning.)

b. *Guess–check–revise*. (*Hint:* Guess $20 and check it. Revise your guess.)

c. *Use a specific case* and *write an equation*. (*Hint:* Choose a specific amount that Donna could have earned and go through the operations used to get the final amount. Then use what you learned to write and solve an equation, letting $n =$ the amount Donna earned and $\frac{1}{2}n =$ the amount she put in the savings account.)

Reflect: Which strategy would you use to solve this problem? Why? ■

Problem-solving strategies play an important role in planning and carrying out solutions to problems. Choosing a useful strategy or several strategies is an important problem-solving skill. What follows on the next page is an elementary student's solution to a problem.

Connection to the PreK–8 Classroom

All problem-solving strategies described are used extensively in PreK–8 classrooms. In the primary grades, act it out, make a model, and choose an operation are emphasized. Other strategies, such as draw a diagram, guess–check–revise, and look for a pattern, also may be introduced. As students progress through the middle grades, they are given many opportunities to use most of these strategies when they solve problems. Certain strategies, such as write an equation, work backward, use a specific case, and use reasoning, are emphasized in grades 7 and 8.

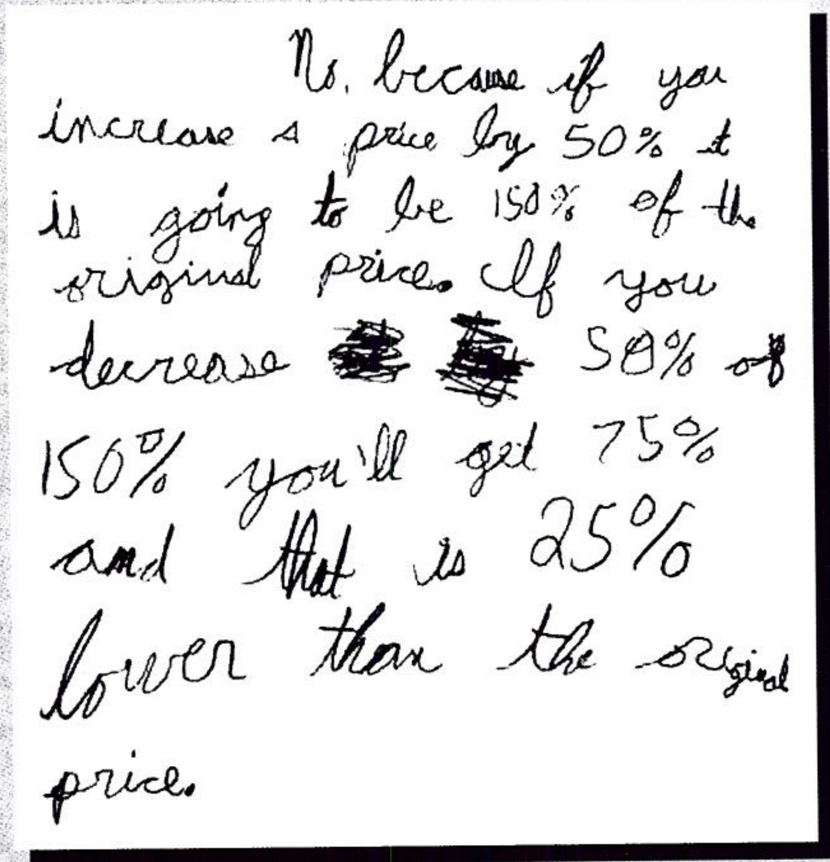
Connection to the PreK–8 Classroom

Analyzing Student Work: Solving a Problem

Below is an elementary student's solution to the following problem:

The price of a necklace was first increased 50% and later decreased 50%. Is the final price the same as the original price? Why or why not?

Consider the student's solution method in light of the strategies identified in the problem-solving model (see Figure 1.6).


No, because if you increase a price by 50% it is going to be 150% of the original price. If you decrease 50% of 150% you'll get 75% and that is 25% lower than the original price.

Source: www.heinemann.com/math, Question_ID = 179, response 3.

Which of the strategies do you think best describes the student's solution process? Justify your choice.

Problems and Exercises for Section 1.3

A. Reinforcing Concepts and Practicing Skills

1. Would you classify the following as a problem for you? Explain. Answers vary.

   Debbie bought a new jacket that originally cost $200. It was discounted 20% plus the store had an additional 10% discount the day Debbie bought the jacket. How much did Debbie pay for the jacket?

2. Why is it impossible to give a prescribed sequence of steps that, if carefully followed, would produce the answer to any problem? Answers vary.

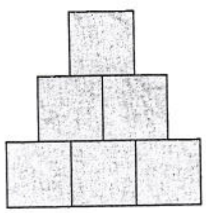
3. Show that the following problem may be solved in more than one way by using the methods in parts (a) and (b):

   A store clerk wanted to stack 21 boxes in a window display in a triangular arrangement like the one shown below. How many boxes should the clerk place on the bottom row? ☛

   a. Solve by extending a pattern.
   b. Solve by drawing a picture.

4. Show that some problems have more than one answer by giving at least two answers to the following problem.

   What change would a restaurant cashier give someone for $1 if he used no more than four of any coin and no coin smaller than a nickel or bigger than a quarter? Answers vary.



5. Consider the following problem:

   Tim said that he lost five coins, none of which was a half dollar, totaling 75¢. What coins might Tim have lost?

   How many solutions does the problem have? ☛
   a. One
   b. More than one
   c. None

   Explain your answer.

6. What are four key stages in the problem-solving process? ☛

In Exercises 7–12, indicate which problem-solving strategy or strategies might be used to solve the problem. You need not actually solve the problem.

7. Chris is 4 years older than her brother Jack. If the sum of their ages is 30, how old is each person? ☛

8. Bella is in a race with 20 other racers. Bella is fifth from last when the race begins. At the end of the race she is third. How many racers did she pass? ☛

9. On the first day, Joe joined a secret club as its only member. Each day after that, one more member joined than on the previous day. What was the membership of the club after 40 days? ☛

10. How many different arrangements of two letters can you make from the letters in the word *problem*, if no repetition of letters is allowed? ☛

11. Anton, Beth, Cal, and Diedre were hired as coaches for basketball, soccer, volleyball, and swimming. Anton's sister was among those hired, and she coaches soccer. Anton does not coach basketball. Beth coaches a water-sport. Which sport was each person hired to coach? ☛

12. Pam opened a checking account with $300. The bank charged $5 for opening the account. She then wrote checks for $20, $50, $75, and $15. After making a deposit, her balance was $200. How much did Pam deposit in her account? ☛

13. Indicate whether an exact answer or an estimate is called for in each of the following questions, and then justify your decision: Answers vary.
    a. How much paint do I need in order to paint my living room?
    b. How many seconds will a swimmer need to swim 100 meters freestyle in an Olympic Games race?
    c. How much would a vacation to Bermuda cost?

14. What information would you need in order to estimate the cost of buying and using a wireless phone in your car for a year? Answers vary.

B. Deepening Understanding

15. The distance around a rectangular room is 44 feet, and 120 tiles (each one is 1 square foot in size) are needed to cover the floor of the room. What are the room's dimensions? Describe the strategy you used to solve the problem. 12 ft by 10 ft.

16. Solve the following problem and describe the strategy that you used:

    Two 2-digit numbers have the same digits. Their sum is 77 and their difference is 27. What are the numbers? The two numbers are 52 and 25.

17. A chairperson and a secretary are to be chosen among four members of a board of directors. How many different possibilities are there for selecting a chairperson and a secretary? 12 possibilities

18. What is the largest sum of money—all in coins but no silver dollars—that a person could have without being able to give change for Answers vary.
    a. a dollar? $1.19
    b. a half dollar? 69 cents
    c. a quarter? 24 cents
    d. a dime? 9 cents
    e. a nickel? 4 cents

    What strategy did you use to solve this problem?