might use *deductive reasoning* to prove that an odd number times an even number is an even number, based on assumptions and the rules of logic.

If you think of mathematics as a process, these two kinds of reasoning are fundamental to doing mathematics. There are two other types of reasoning that will be discussed in later chapters: proportional reasoning (Chapter 7) and spatial reasoning (Chapters 10 and 11). We will now consider inductive reasoning and recognizing patterns in some detail.

Inductive Reasoning and Patterns

Inductive Reasoning. Mini-Investigation 1.4 sets the stage for understanding inductive reasoning.

Talk about situations in which you have used a similar type of reasoning.

MINI-INVESTIGATION 1.4 Using Mathematical Reasoning

What is the main characteristic of the type of reasoning used in the cartoon?

"Water boils down to nothing . . . snow boils down to nothing . . . ice boils down to nothing . . . everything boils down to nothing."

Cartoon by Ed Fisher; © 1966, The Saturday Review, Inc. Reprinted by permission of Ed Fisher.

The preceding cartoon contains a somewhat humorous use of *inductive reasoning*, a process we more generally describe as follows.

Description of Inductive Reasoning

Inductive reasoning involves the use of information from specific examples to draw a general conclusion. The general conclusion drawn is called a **generalization.**

We can further describe inductive reasoning in the following way: After watching an event that gives the same results several times in succession, an observer detects a pattern or relationship and tentatively concludes that the event will always have the same outcome. For example, suppose you move into a new apartment and see a neighbor leave to walk her dog at 7 A.M. on Monday, again at 7 A.M. on Tuesday, and so on for the rest of the week. Based on your observations, you form a generalization and conclude, "My neighbor always leaves at about 7 A.M. to walk her dog." We can use this example to illustrate the steps in the inductive reasoning process.

Procedure for Using the Inductive Reasoning Process

■ Check several examples of a possible relationship.	You observe your neighbor as she leaves to walk her dog each day for a week.
■ Observe that the relationship is true for every example you checked.	You observe that each day she leaves at 7 A.M. to walk her dog.
■ Conclude that the relationship is *probably* true for all other examples and state a generalization.	You conclude, "My neighbor always leaves at 7 A.M. to walk her dog."

To preview some ways in which we use the inductive reasoning process in this book, let's consider some mathematical situations. First, we double a number and add 1 each time. Then we use inductive reasoning to form a generalization.

Examples: $2 \times 3 + 1 = 7$; $2 \times 6 + 1 = 13$; $2 \times 7 + 1 = 15$; $2 \times 10 + 1 = 21$.

Generalization: When we double any number and add 1, the result is an odd number.

Next, let's triple a number and add 1 each time. Then we use inductive reasoning to form a generalization.

Examples: $3 \times 4 + 1 = 13$; $3 \times 6 + 1 = 19$; $3 \times 10 + 1 = 31$; $3 \times 12 + 1 = 37$.

Generalization: When we triple any number and add 1, the result is an odd number.

We can also do some calculations with the assistance of technology. Then we use inductive reasoning to form a general conclusion.

Examples: The screen shows several calculations carried out with a graphing calculator.

```
-2×7
                 -14
-5×8
                 -40
-1×11
                 -11
```

Generalization: The product of a negative number and a positive number is a negative number.

Although the generalization that doubling and adding 1 always gives an odd number is true, we can produce another example, $3 \times 5 + 1 = 16$, which shows that the generalization that tripling and adding 1 always gives an odd number is false! Thus the use of inductive reasoning to form a generalization based on specific examples cannot ensure that the generalization will hold true for all possible cases. An example that disproves a generalization is of central importance to inductive reasoning and is called a *counterexample.*

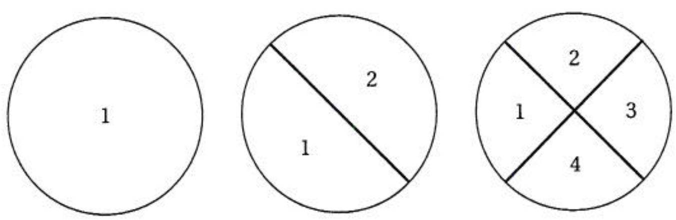
Description of Counterexample

A **counterexample** is an example that shows a generalization to be false.

Example 1.2 further illustrates the need to look for a counterexample when you use inductive reasoning, and it presents some geometric situations that may surprise you!

Example 1.2 Finding a Counterexample

Use inductive reasoning to form a generalization from the following examples:



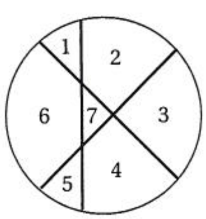
Number of chords	Maximum number of regions
0	1
1	2
2	4

Look for a counterexample that might prove the generalization false.

SOLUTION

Generalization: The number of regions formed by drawing chords in a circle doubles in each successive example.

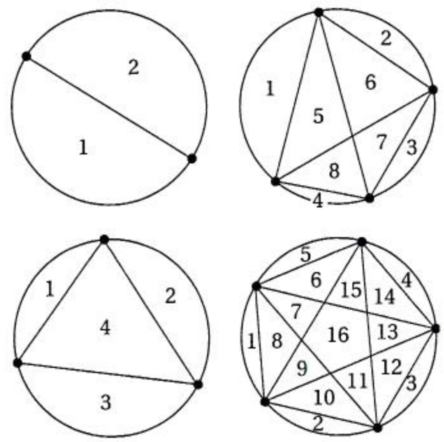
Counterexamples: When we draw three chords in a circle, the maximum number of regions formed is seven, which is not the double of four regions.



YOUR TURN

Practice: Use inductive reasoning to form a generalization from the table on the next page. Then draw a large, 3-inch-diameter circle with six points on it to complete the last row of the table. Look for a counterexample that might prove the generalization false.

Reflect: Suppose you found that a generalization was correct for the first 100 examples. Does this prove the generalization true? Why or why not? ■



Number of points on the circle	Maximum number of regions
2	2
3	4
4	8
5	16
6	?

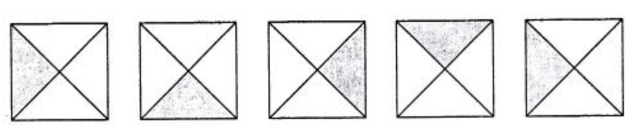
In Example 1.2, you could have formed the generalization that the maximum number of regions doubles each time and that the number of regions for six points would therefore be 32. The counterexample that can be found for this generalization shows that you must exercise care when using inductive reasoning. Clearly, you don't know whether a generalization formed by inductive reasoning is true or false. If you look at a lot of examples and can't find a counterexample for a generalization, you may conclude that the generalization probably is true, but you can't be sure until you have proved the generalization.

Patterns. Mathematics is sometimes defined as the science of studying patterns. When forming generalizations by inductive reasoning, you used patterns discovered by looking at several examples. Sometimes, as when you were looking for a pattern in the joining of midpoints of the sides of several quadrilaterals, the order of the examples didn't make any difference. At other times, as when you were looking at the number of regions formed by connecting points on a circle, as in Example 1.2, the order of the examples was crucial in helping you discover a pattern. In this section, we work primarily with ordered patterns.

Sequences. A pattern involving an ordered arrangement of numbers, geometric figures, letters, or other entities is called a **sequence.** The numbers, geometric figures, or letters that make up a sequence are called the **terms of the sequence.** The first four terms of a familiar numerical sequence are

1, 3, 5, 7, _?_, _?_, _?_.

The pattern is obvious, and we can easily extend the sequence by giving the next several terms. We can also look for patterns in sequences of geometric figures such as



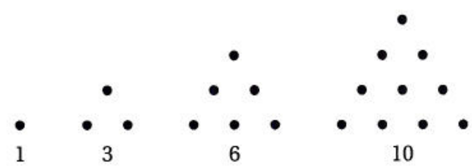
???

Discovering the pattern of rotation of the figure lets us anticipate the next several terms in the sequence.

Numerical sequences may be classified according to the methods used to find their terms. For example, a numerical sequence in which each term is obtained from the previous term by *adding* a fixed number is called an **arithmetic sequence.** The fixed number is called the **common difference** for the sequence. For example, the odd numbers form an arithmetic sequence with common difference 2; or 1, 3, 5, 7, 9, 11, ...

A numeric sequence in which each term is obtained from the previous term by *multiplying* by a fixed number is called a **geometric sequence.** The fixed number is called the **common ratio** for the sequence. An example of a geometric sequence with common ratio 2 is 1, 2, 4, 8, 16, 32, ...

The numbers of dots in the following triangle dot arrays are an example of a sequence that is neither arithmetic nor geometric:



Note that for the dot array sequence, 2 is added to the first term to get the second; 3 is added to the second term to get the third; 4 is added to the third to get the fourth; and so on. You observe a pattern of increases, and inductively conclude that 5 would be added to the fourth term to get the fifth term. Example 1.3 extends the ideas of arithmetic and geometric sequences.

Example 1.3 Extending Sequences

Determine whether the sequence is arithmetic, geometric, or neither. If one exists, give the common difference or common ratio and then give the next three terms in the sequence.

a. 1, 3, 9, 27, 81, . . .
b. 2, 5, 8, 11, 14, . . .
c. 1, 4, 9, 16, 25, . . .

SOLUTION

a. We see a pattern: Each term is obtained by multiplying the term before it by 3, so the sequence is geometric, with common ratio 3. The next three terms are 243, 729, and 2,187.

b. We see a pattern: Each term is obtained by adding 3 to the term before it, so the sequence is arithmetic, with common difference 3. The next three terms are 17, 20, and 23.

c. The pattern isn't as obvious. Succeeding terms aren't obtained by either adding or multiplying preceding terms, so the sequence is neither arithmetic nor geometric. Upon further investigation, we see that the next three terms are the perfect squares 36, 49, and 64.

YOUR TURN

Practice: Determine whether the sequence is arithmetic, geometric, or neither. If one exists, give the common difference or common ratio and then give the next three terms in the sequence.

a. 10, 20, 30, 40, . . .
b. 5, 5, 10, 10, 15, . . .
c. 1, 4, 16, 64, . . .

Reflect: Which type of sequence, arithmetic or geometric, increases faster? Explain. ■

To extend the ideas of arithmetic and geometric sequences, let's consider a general way to represent their terms. For example, suppose that a is the first term of an arithmetic sequence and that d is the common difference. We *add* the common difference to the first term to get the second term, so we can represent the second term as $a + d$. Adding d again, we obtain the third term, $a + d + d$, or $a + 2d$. Thus, the first several terms of an arithmetic sequence may be written as

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

Using inductive reasoning, we note that, because the second term is $a + 1d$, the third term is $a + 2d$, and the fourth term is $a + 3d$, the nth term of an arithmetic sequence is $a + (n - 1)d$.

We may represent a geometric sequence in a general way by using a similar line of reasoning. For example, suppose that a is the first term of the sequence and that r is the common ratio. We *multiply* the common ratio by the first term to get the second term, so we can represent the second term as ar. Multiplying by r again, we note that the third term is $a \times r \times r$, or ar^2. Thus, the first several terms of a geometric sequence may be written as

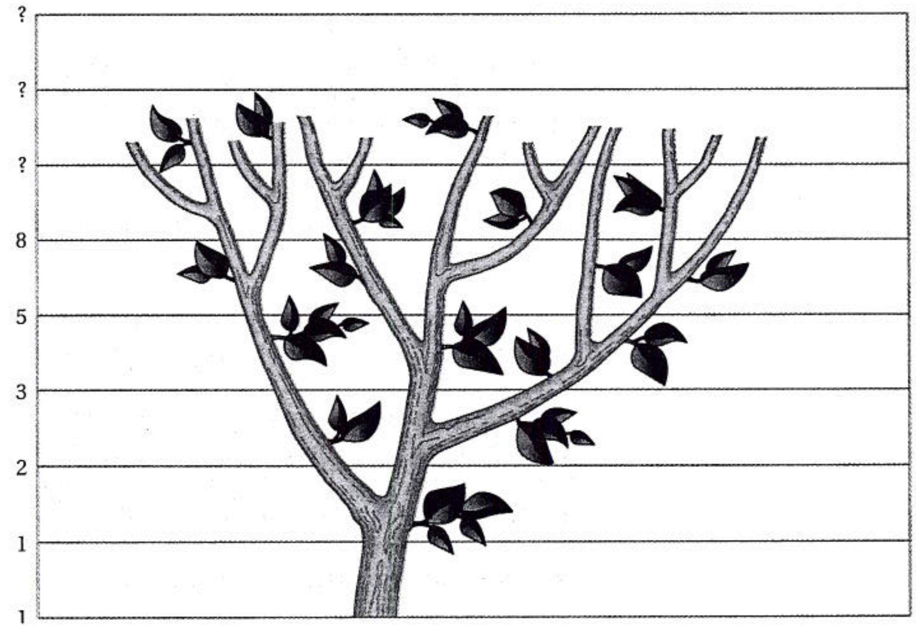
$$a, ar, ar^2, ar^3, ar^4, \ldots$$

Using inductive reasoning, we note that as the second term is $a(1r)$, the third term is ar^2, and the fourth term is ar^3, the nth term of a geometric sequence is ar^{n-1}. In Exercise 77 on p. 38, you will use these ideas to find a specified term of a sequence when you are given some terms and the common difference or ratio.

Often, as in the B.C. cartoon in Figure 1.5 showing the effect of inflation on service charges, we associate a sequence of numbers with events, objects, or relationships between objects. In B.C.'s cartoon world, as inflation increases, will the next change be 6 clams, or will it be 8 clams? When you use inductive reasoning to discover a pattern in these numbers, you have discovered a regularity in a "real-world" situation.

FIGURE 1.5
The beginning of a sequence.

Patterns in Real-World Data Tables. Real-world data are often recorded in a **table of values,** which shows relationships between different data categories. Often two types of patterns occur in such a table: *column extension* and *row relationship patterns.* Consider the following table of values created from this picture showing the way tree branches sometimes grow:



Level Number	Number of Branches
1	1
2	1
3	2
4	3
5	5
6	8

This table of values can be extended by discovering a *column extension pattern* in which each digit in the right-hand column is found by adding the two preceding digits in the column. Using this column extension pattern, we determine that the next digit in the column will be $5 + 8$, or 13. The sequence shown in the right-hand column of the table is known as a Fibonacci sequence. It was named after Leonardo Pisano (also called Fibonacci; 1170–1230), who discovered it. In a **Fibonacci sequence,** each term is the sum of the two preceding terms.

Now consider the data in the following table of values of predicted distances for travel on a bicycle at a predetermined rate of speed. This table illustrates how it is sometimes useful to also discover a *row relationship pattern*; in this case, in each row the distance is always nine times the number of hours. Using this row relationship pattern, we determine that in the next row of the table the time will be 5 hours and the distance will be 45 miles. Note that this table can also be extended by discovering a column extension pattern. The numbers in the right-hand column form

Time in Hours (t)	Distance in Miles (d)
1	9
2	18
3	27
4	36
.	.
.	.
.	.
.	.
t	?

an arithmetic sequence with common difference 9. So, to extend the right-hand column, we simply add 9 and 36 to get 45.

Although column extension patterns are useful, using a row relationship pattern is often more efficient. For example, to find the distance for a 30-hour bike travel time, we would have to extend the right column in the table to include 26 more numbers. However, once we have found the row relationship pattern, we can simply multiply 30 by 9 to obtain the corresponding distance, 270 miles.

Example 1.4 gives further examples of real-world situations in which patterns may be used to discover a relationship between the numbers in a table of values.

Example 1.4 Problem Solving: Ramp Racer

A maker of miniature racing cars timed how far his prized car rolled down a ramp in various lengths of time and recorded the following information:

Length of Time in Seconds (s)	Distance in Feet (d)
1	1
2	4
3	9
4	16
.	.
.	.
.	.
s	?

Look for a pattern in the table and determine the distance the miniature car rolled down the ramp in 8 seconds.

SOLUTION

Jeff's thinking: I looked at the right-hand column and noticed that the distances increased from 1 to 4, 4 to 9, and 9 to 16, or 3, 5, and 7 feet. So I continued this column pattern and increased 16 by 9 to get 25, increased 25 by 11 to get 36, increased 36 by 13 to get 49, and increased 49 by 15 to get 64. The car traveled 64 feet down the ramp in 8 seconds.

Grenada's thinking: I looked for a relationship between the numbers in each row. I noticed that 1 is 1 squared, 4 is 2 squared, 9 is 3 squared, and so on. So for 8 seconds, I used the same pattern to find that the distance would be 8 squared, or 64. The car traveled 64 feet down the ramp in 8 seconds.

YOUR TURN

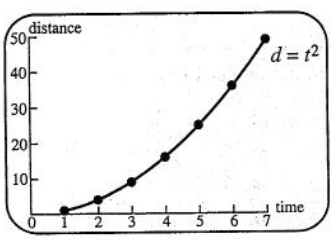
Practice: A health spa gave out the following table to show the weight loss per week that a 166-pound person could expect from using the spa's facilities:

Number of Weeks	Weight (lb.)
1	164
2	162
3	160
4	158

Look for a pattern in the table and determine the 166-pound person's weight after 8 weeks of workouts and dieting at the spa.

Reflect: Which pattern, the column extension or the row relationship, do you think is the most efficient way to solve the example problem? Why? ■

A graphing calculator can also provide visual information about the pattern of numbers in a table of values. For example, to solve the Ramp Racer problem in Example 1.4, we can show the points that represent the pairs of numbers from the table on a graph in which the x-axis represents the time (t) and the y-axis represents the distance (d). We scale the x-axis in 1-second intervals and the y-axis in 10-foot intervals and then enter the data to produce the screen in the margin.



The graph does not show a pattern of constant change. That is, when the number of seconds increases by 1, the *distance* does not always increase by a constant amount. By magnifying portions of the graph on the screen, we can see that the *change in the change in distance* is constant and increases by 2 feet for each second of time increase.

Patterns in Sequences of Number Sentences. Not only can you discover patterns from sequences of numbers, you also can discover patterns from a sequence of number sentences. For example, consider the following sequence of number sentences involving sums of consecutive whole numbers:

$$1 + 2 = \frac{2 \times 3}{2}, \quad \text{or } 3;$$

$$1 + 2 + 3 = \frac{3 \times 4}{2}, \quad \text{or } 6;$$

$$1 + 2 + 3 + 4 = \frac{4 \times 5}{2}, \quad \text{or } 10;$$

$$1 + 2 + 3 + 4 + 5 = \frac{5 \times 6}{2}, \quad \text{or } 15;$$

$$1 + 2 + 3 + 4 + 5 + \cdots + n = ?$$

Study the pattern in these number sentences. Use inductive reasoning, observing that the last number to be added is the key to calculating the sum. If you multiply this last number by the number immediately following it and then divide this product by 2, you get the sum. Exercise 80, p. 38, asks you to look for a pattern in another sequence of number sentences.

Deductive Reasoning

We now consider a type of mathematical reasoning, called *deductive reasoning*, that is used in drawing logical conclusions and in presenting convincing arguments or proofs. This section will focus on the following aspects of deductive reasoning:

- understanding statements and negations;
- understanding conditional (if–then) statements and deciding when they are true;
- using rules of logic—affirming the hypothesis and denying the conclusion;
- writing and analyzing converse, inverse, and contrapositive statements; and
- understanding and using conjunctions, disjunctions, and biconditional statements.

Statements and Negations. A basic part of deductive reasoning involves the use of statements. Sometimes people think statements and sentences are the same. But we will make a distinction between them. Consider the following sentences:

a. $x + y = 10$.

b. $3(x + y) = 3x + 3y$.

c. George Washington was our 5th president.

d. George Washington was our 1st president.

e. The product of two numbers is odd.

Although (a)–(e) are all sentences, mathematicians would classify only sentences (b), (c), and (d) as statements. A statement is a sentence that can be determined to be true or false. Sentence (a) fails to achieve this because we do not know the values associated with x and y. Sentence (e) fails to be a statement because we do not know what the two numbers are. This suggests the following definition:

Definition of a Statement

A **statement** is a sentence that is either true or false but not both.

The negation of a statement is a statement that has the opposite truth value of the given statement. This can be expressed in the following way.

Definition of a Negation

The **negation** of a statement p is the statement not p (denoted $\sim p$). If p is true, then $\sim p$ is false. If p is false, then $\sim p$ is true.

The following are examples of statements and their negations.

r: The average of 2, 3, and 4 is 3.
$\sim r$: The average of 2, 3, and 4 is not 3.
s: $4^2 = 16$
$\sim s$: $4^2 \neq 16$
t: Jennifer Aniston was once married to Brad Pitt.
$\sim t$: Jennifer Aniston never married to Brad Pitt.

Example 1.5 illustrates the difference between sentences and statements.

Example 1.5 Analyzing Statements and Negations

Determine if the following sentences are statements. If so, form their negations. If not, say no negation exists.

a. The sum of two even numbers is even.
b. The sum of two odd numbers is odd.
c. The square of a number is greater than 10.
d. A parallelogram has four sides.

SOLUTION

a. The sum of two even numbers is not even (or is odd).
b. The sum of two odd numbers is not odd (or is even).
c. No negation exists.
d. A parallelogram does not have four sides.

YOUR TURN

Practice: Write a false statement and its negation. Is the negation true or false? Explain.

Reflect: Suppose a given statement is true. Is the negation of its negation true or false? Explain. ■

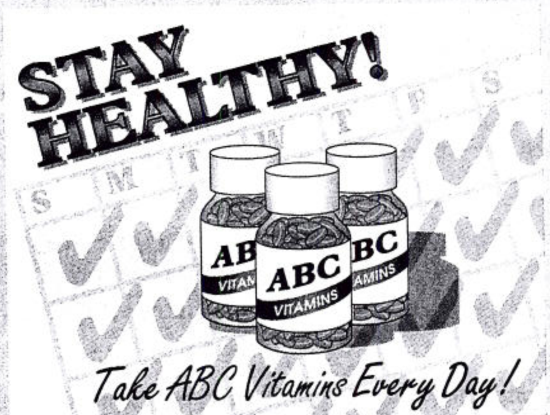
Conditional Statements. A particular type of statement that is used in deductive reasoning is an "if–then" statement. Mini-Investigation 1.5 illustrates this type of statement.

In mathematics, statements in *if–then* form are called **conditional statements.** The *if* part of a conditional statement is called the **hypothesis,** and the

Write an if–then statement about one of your favorite products.

MINI-INVESTIGATION 1.5 **Using Mathematical Reasoning**

What statement in the form **If ..., then....** could you write about the products in the following advertisement?



then part is called the **conclusion.** These ideas are illustrated in the following statement:

Hypothesis: If you wear Super Shoes, — Conclusion: then you will play like a champion.

If you wear Super Shoes, then you will play like a champion.

Example 1.6 shows how to decide whether a conditional statement is true or false.

Example 1.6 **Analyzing Conditional Statements**

Suppose that your basketball coach made the following conditional statement:

If you play well in practice, then you will start in tomorrow's game.

In which of the following cases would you feel that you were being treated unfairly and that the coach didn't tell the truth?

Case 1: You play well in practice (hypothesis true).
You start the game (conclusion true).

Case 2: You play well in practice (hypothesis true).
You do not start the game (conclusion false).

Case 3: You do not play well in practice (hypothesis false).
You start the game (conclusion true).

Case 4: You do not play well in practice (hypothesis false).
You do not start the game (conclusion false).

SOLUTION

Case 2 is the only instance in which the coach did not tell the truth.

YOUR TURN

Practice: Suppose that a friend made the following conditional statement:

If I go on the trip, then I will bring you back a T-shirt.

Describe the conditions under which your friend would not have told the truth.

Reflect: Explain why the coach's statement in the example problem is true in cases 1, 3, and 4. ■

In Example 1.6, you may have discovered the following procedure for deciding whether a conditional statement is true or false.

Procedure for Deciding Whether a Conditional Statement Is True or False

1. Decide whether the hypothesis and the conclusion are true or false.
2. Use the following to decide if the statement is true or false:
   a. When both the hypothesis and conclusion are true, the conditional statement is true.
   b. When the hypothesis is true and the conclusion is false, the conditional statement is false.
   c. When the hypothesis is false and the conclusion is true, the conditional statement is true.
   d. When both the hypothesis and conclusion are false, the conditional statement is true.

Truth Table for Conditional Statements. We can summarize the information above in a different way using a *truth table.* In doing so, we will use the following definition of a conditional statement.

Definition of a Conditional Statement

A statement that can be written in the form "if. . ., then . . ." is called a **conditional statement.** Conditional statements are denoted symbolically by writing $p \rightarrow q$.

Suppose we define p and q of a conditional statement in the following way:

p: U.S. Women's soccer team scores at least two goals

q: U.S. Women's soccer team wins the match

Now consider the following conditional statement,

If the U.S. Women's soccer team scores at least two goals, then they will win.

which can be represented symbolically by $p \rightarrow q$ (read "p implies q"). The truth table in the margin can be created.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The only time the statement $p \rightarrow q$ is considered false is when p is true and q is false. In our case this would mean that the soccer team scored at least 2 goals but did not win the match. Although it may seem strange, logicians have decided that if the hypothesis (p) of a statement is false, then the conditional statement $p \rightarrow q$ is considered true regardless of whether q is true or false.

Rules of Logic. We now consider two rules of logic that are frequently used in deductive reasoning. Let's first look at logic rule A, which is used when both a conditional statement and its hypothesis are true.

Logic Rule A

- If p, then q (true) → If Lee's score is higher than 94 (p), then he will get an A (q).
- p is true → Lee's score is higher than 94.

- Therefore q is true → Lee got an A.

The letters p and q in the rule represent the hypothesis and conclusion, respectively. Logic rule A allows you to conclude that the conclusion is true. Logic rule A is sometimes called **affirming the hypothesis** because it is used when the given hypothesis is true. This form of deductive reasoning is also called **modus ponens.**

Logic rule B is used when a conditional statement is true and its conclusion is false.

Logic Rule B

- If p, then q (true) → If Dana stays home (p), then she will miss her class (q).
- q is false → Dana did not miss her class.

- Therefore p is false → Dana did not stay home.

Logic rule B allows you to conclude that the hypothesis is false. Logic rule B is sometimes called **denying the conclusion** because it is used when the given conclusion is false. This form of deductive reasoning is also called **modus tollens.** Example 1.7 provides additional insight into the logic rules A and B.

Example 1.7 Using Rules of Logic

What conclusion can be drawn from the following conditional statements?

a. If today is Saturday, then we play the big game.
We do not play the big game.

b. If all sides of a quadrilateral are the same length, then the quadrilateral is a rhombus.
All sides of square $ABCD$ are the same length.

Identify which rule of logic is used for each.