- 37. Use the four elements a, b, c, and d and generate an operation, #, that has all four addition properties. Describe the operation in a table like that in Exercise
- 38. Return to the picture you drew in Mini-Investigation 2.5 about how addition and subtraction are different and how addition and subtraction are alike. What changes would you make to this diagram now? Explain why. Auswors vary
- 39. Return to the definition of less than for whole numbers.
 - **a.** Why is k > 0 an important part of the definition? What would happen if it were omitted?
 - b. How would a definition of greater than based on whole-number operations and the properties of these operations differ from the definition of less than?
- D. Communicating and Connecting Ideas
- 40. Because subtraction is defined by using addition, you could argue that subtraction isn't an operation in its own right-that all joining, separating, and comparing situations could be described with an addition equation.

- With two or three other classmates, present an argument orally or in writing to support subtraction as an operation having its own definition and symbolism.
- 41. Historical Pathways. When recording on papyrus, the Egyptians used a set of mathematical symbols that included separate symbols for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and so on. For example, the symbol for 432 was written with the symbols for 400, 30, and 2 lined up next to each other. Addition with these symbols probably required the use of extensive addition tables showing the sums of each pair of symbols. How might the scribes have dealt with subtraction?
- 42. Making Connections. For many pairs of whole numbers, a and b, no whole number represents a - b. That is, whole numbers are not closed under the operation of subtraction. Based on your earlier experiences in mathematics, describe the kinds of numbers that, when included with whole numbers, bring closure to subtraction. What physical situations could these numbers model? How would these numbers be related to the set of whole numbers? How could these numbers be helpful when you use subtraction to compare two whole numbers?

Section \mathbb{A}^{S}

Multiplication and Division of Whole Numbers

Using Models and Sets to Define Multiplication 🔞 Properties of Multiplication Modeling Division Using Multiplication to Define Division - Comparing Division to Multiplication - The **Division Algorithm for Whole Numbers**

In this section, we use models to help explain multiplication and division of whole numbers and the actions involved in these operations. We also examine the mathematical definitions and properties of the operations of multiplication and division and investigate the relationship between these operations.

Essential Understandings for Section 2.3

- Some real-world problems involving joining equal groups, separating equal groups, comparison, or combinations can be solved using multiplication; others can be solved using division.
- Multiplying by x is the inverse of dividing by x (when $x \neq 0$).
- Properties of whole numbers apply to certain operations but not others.
- Two numbers can be multiplied in any order.
- The product of any nonzero number and 1 is the number.
- Three or more numbers can be grouped and multiplied in any order.

Dra

Mini-Investigation 2.7 asks you to compare the operations of multiplication and division.

Draw a picture to illustrate your comparison of multiplication and division.

MINITINVESTIGATION 2.7

Making a Connection

How are multiplication and division different, and how are they alike?

Using Models and Sets to Define Multiplication

Models for multiplication are similar to models for addition. Addition can be modeled by joining two sets, and multiplication can be modeled by joining a certain number of equivalent sets. When these sets are arranged in equal rows and columns, as shown in Figure 2.20, this arrangement of the three equivalent sets is called a **rectangular array.** In any multiplication model involving discrete objects, one number represents the number of sets (rows) and one number represents the number of objects in each set (row). The idea behind these models and the language of sets is used in the following description of multiplication:

 $a \times b$ is the number of elements in the union of a equivalent sets, each containing b elements.

Addition also can be modeled by joining two directed segments along the number line, and multiplication can be modeled by joining a certain number of directed segments of equal length, as illustrated in Figure 2.21. In any multiplication model involving lengths, one number represents the number of segments being joined and one number represents the length of one segment.

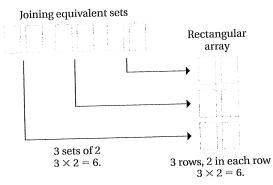


FIGURE 2.20 Multiplication as joining equivalent sets and as a rectangular array.

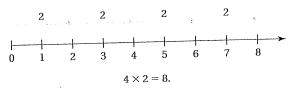


FIGURE 2.27

Multiplication as joining segments of equal length on a number line.

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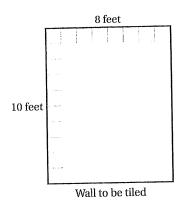
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 $10\times 8=80.$ FIGURE 2.22 Multiplication as the area of a rectangle.

Area Model. In an **area model** of whole-number multiplication, the two numbers being multiplied represent the dimensions of a rectangle. The area of the rectangle is the result of the multiplication. For example, if you are decorating 1-foot-square tiles to cover a rectangular wall that is 10 feet high and 8 feet wide, you naturally ask, "How many tiles do I need?" When you answer this question, you are demonstrating a fundamental meaning of multiplication, as illustrated in Figure 2.22.

The area model is similar to the array model because the area of a rectangle with whole-number length and width can be thought of as determining the total number of unit squares when the number in each row and the number of rows are known. However, the area and array models are different in that the rectangular dimensions aren't always whole numbers and in that the array model deals with discrete objects, whereas the area model deals with a continuous region.

Using Repeated Addition to Define Multiplication. The idea that 5×3 , for example, can be found by putting together 5 sets with 3 objects in each set suggests that 5×3 can be interpreted as 3 + 3 + 3 + 3 + 3. Thus we sometimes say that multiplication is *repeated addition*.

Bjanglifolg of Militiplication virginasted Addition

In the **multiplication of whole numbers**, if there are m sets with n objects in each set, then the total number of objects $(n + n + n + \cdots + n)$, where n is used as an addend m times) can be represented by $m \times n$, where m and n are **factors** and $m \times n$ is the **product**.

Using the Language of Sets to Define Multiplication. As described earlier, we can use set language to complete a description of $a \times b$ as the number of elements in the union of a disjoint equivalent sets, each containing b elements. Although this description provides a possible definition of multiplication of whole numbers, let's consider another set idea that can be used to define multiplication, called the *Cartesian product* of two sets. An example of the idea of an **ordered pair** used in the definition is (cat, dog), which is a different ordered pair than (dog, cat).

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The **Cartesian product** of two sets A and B, $A \times B$ (read "A cross B") is the set of all ordered pairs (x, y) such that x is an element of A and y is an element of B.

For example, if $A = \{1, 2, 3\}$ and $B = \{p, q\}$, then $A \times B = \{(1, p), (1, q), (2, p), (2, q), (3, p), (3, q)\}$. Note that sets A and B can be equal. Example 2.11 illustrates the idea of a Cartesian product of a set with itself.

Example 3

Illustrating a Cartesian Product

In a particular game of chance, a player's turn consists of rolling a die twice. What are the possible results a player could get on a turn? How many results are there?

SOLUTION

A player could roll a 1 first, then a 1, 2, 3, 4, 5, or 6 on the second roll. Similarly, if 2 came up on the first roll, the second roll could be 1, 2, 3, 4, 5, or 6. The entire set of possibilities may be represented in the diagram on p. 91. Each die can be

Unde

modeled by a set of six numbers, $S = \{1, 2, 3, 4, 5, 6\}$. The 36 resulting pairs of numbers represent the Cartesian product, $S \times S$.

Number on second roll

			and the second				
		1	2	3	4	5	6
ē	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
first r	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
on fii	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
Number o	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
Ž						(6, 5)	

YOUR TURN

Practice: In another game, a player's turn consists of two spins of a spinner, with options 0, 1, 2, or 3 on the spinner. What are the possible results a player could get on a turn? How many results are there?

Reflect: What generalization can you discover from the two problems in this example?

We now use the idea of Cartesian product to give a commonly used definition of multiplication of whole numbers.

Definition of Multiplication of Whoir Number

In the **multiplication of whole numbers**, if A and B are finite sets with a = n(A) and b = n(B), then $a \times b = n(A \times B)$. In the equation $a \times b = n(A \times B)$, a and b are called **factors** and a and a is called the **product**.

The essential idea of the definition is that $a \times b$ is the number of elements in the Cartesian product of two sets, one set containing a elements and the other set containing b elements. Multiplication as the representation of a Cartesian product can occur in a variety of physical situations. As illustrated by Example 2.12, they include counting combinations and rectangular arrays. Useful problem-solving strategies include "act it out" or "draw a diagram."

Example

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Problem Solving: Color Combinations for Invitations

Suppose that you are using construction paper to make invitations for a club function. The construction paper comes in blue, green, red, and yellow, and you have gold, silver, or black ink. How many different color combinations of paper and ink do you have to choose from?

Working Toward a Solution

Understand the problem

What does the situation involve?	Selecting colors of paper and colors of ink for invitations.
What has to be determined?	The number of different combinations.
What are the key data and	Four colors of paper; three colors of ink.
conditions?	

Develop a pla	lar	1
---------------	-----	---

What are some assumptions? What strategies might be useful? Are there any subproblems?

or calculated?

What method of calculation should be used?

Implement the plan

Should the answer be estimated

How should the strategies be used?

Each invitation combines one color of paper and one color of ink.

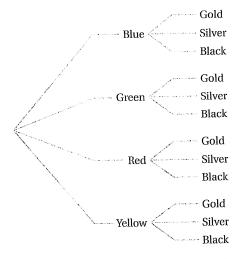
Act it out or draw a diagram.

Find the number of different invitations that can be made with blue

Calculate the exact number.

Use the diagram to count.

Use a choice, or tree, diagram to match each possible choice of paper with each choice of ink.



Look back

What is the answer?

Is the interpretation correct?

Is the calculation correct? Is the answer reasonable?

Is there another way to solve the problem? Four kinds of paper, each with three colors of ink, make 12 combinations.

Check: The diagram fits the problem.

Check: The count is correct.

Yes. Three choices of ink for each color of paper would triple the kinds of invitations made with one color of ink.

As shown, you can use an array of ordered pairs to list the kinds of invitations. Or you can use a multiplication equation:

 $4 \times 3 = 12$ combinations.

In the array of ordered pairs, each color of paper is matched with each color of ink in an ordered pair that describes an invitation.

	Gold	Silver	Black
Blue	(B, G)	(B, S)	(B, Bk)
Green	(GR, G)	(GR, S)	(GR, Bk)
Red	(R, G)	(R, S)	(R, Bk)
Yellow	(Y, G)	(Y, S)	(Y, Bk)

YOUR TURN

Practice: From a committee, one woman's name will be drawn and one man's name will be drawn to see who serves on a community advisory board. If 9 women and 11 men are on the committee, how many different possible combinations of one woman and one man are there?



Reflect: What solution to the practice problem is suggested by the Cartesian product definition of multiplication?



Although multiplication is usually first introduced in the early grades as joining equivalent sets or as repeated addition, young students need to experience the other actions connected to multiplication. These experiences include looking for all possible pairs, finding the number in an array, and finding the number of tiles in a rectangle. They provide for a broader understanding of the concept of multiplication and its use in counting outcomes for sample spaces in probability and in determining areas and volumes of surfaces and solids.

Properties of Multiplication

As the pirate in the cartoon in Figure 2.23 seems to know, multiplication, like addition, is commutative. The properties that are important in addition—the closure property, the identity property, the commutative property, and the associative property—also hold for multiplication.

HERMAN®



FIGURE 2.23

"Can you change two pieces of eight for eight pieces of two?"

Herman \odot is reprinted with permission from LaughingStock Licensing Inc., Ottawa, Canada. All rights reserved.

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nan's men ns of We summarize the properties of multiplication as follows:

Properties of Multiplication of Whole Number:

Closure property For whole numbers a and b, $a \times b$ is a unique

whole number.

Identity propertyThere exists a unique whole number, 1, such that

 $1 \times a = a \times 1 = a$ for every whole number a. Thus 1 is the *multiplicative identity element*.

Commutative property For whole numbers a and b, $a \times b = b \times a$.

Associative property For whole numbers a, b, and c, $(a \times b) \times c = a \times (b \times c)$

 $(a \times b) \times c = a \times (b \times c).$

Zero property For each whole number a, $a \times 0 = 0 \times a = 0$.

Distributive property of For whole numbers a, b, and c, multiplication over addition $a \times (b + c) = (a \times b) + (a \times c).$

Figure 2.24 shows one way to verify the commutative property of multiplication.

The properties of multiplication different from the properties of addition include a different identity element for the identity property and the zero property. The

a different identity element for the identity property and the zero property. The distributive property of multiplication over addition connects the two operations.

Identity Property of Multiplication. According to the identity property of multiplication, there is a specific whole number, one, that when multiplied by a whole number, n, leaves n as the product:

One is the unique whole number such that for each whole number a, $a \times 1 = 1 \times a = a$.

Thus, 1 is called the *multiplicative identity element*.

Zero Property of Multiplication. The **zero property of multiplication** states that, whenever 0 is used as a factor, the product is always 0:

For each whole number a, $a \times 0 = 0 \times a = 0$.

This property is important because 0 can often hide the value of another number in multiplication. For example, in the equation $a \times b = 0$, a could be any number if b = 0. The zero property of multiplication also leads to an important conclusion about division developed on p. 99.

Distributive Property of Multiplication over Addition. The distributive property of multiplication over addition connects the operations of multiplication and addition. In general,

For whole numbers
$$a$$
, b , and c , $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$.

This property may be used to generate unknown multiplication facts from known ones and may be represented by looking at an array model from two different viewpoints, as shown in Figure 2.25.

Analyzing a Textbook Page: The Commutative Property



The commutative property of multiplication is a useful property for learning related basic facts. The following page in an elementary textbook shows a real-world model for the commutative property of multiplication.

Activity

Does it matter in what order you multiply?

Lynn multiplies two different ways to find how many squares are in her quilt.

First she holds her quilt like this.



$$\begin{array}{c} X X X X X \\ X X X X X \\ X X X X \end{array}$$

$$\begin{array}{c} 3 \text{ rows} \\ 4 \text{ in each row} \\ 4 + 4 + 4 = 12 \\ 3 \times 4 = 12 \end{array}$$

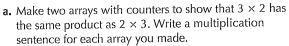
Then she holds the quilt like this.



There are 12 squares in Lynn's quilt.

The Commutative (order) Property of Multiplication says you can multiply in any order and the product is the same.

So,
$$3 \times 4 = 4 \times 3$$
.



b. How do the arrays below show the Commutative Property of Multiplication?



FIGURE 2.24

Source: Scott Foresman–Addison Wesley Mathematics, Grade 3, p. 263. © 2004 Pearson Education. Reprinted with permission.

What other real-world models of the commutative property of multiplication can you think of?

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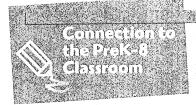
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 $3 \times (5+1) = (3 \times 5) + (3 \times 1).$

FIGURE 2.25

Array model of the distributive property of multiplication over addition.



The identity property, commutative property, zero property of multiplication, and the distributive property of multiplication over addition may be used to help students learn multiplication facts. Of the 100 products with factors of 0-9, 19 have a factor of 0, and 17 more have a factor of 1. Seeing the patterns caused by the zero and identity properties automatically allows students to know 36 of the basic multiplication facts. Of the other 64 products, 8 are doubles (a imes a), leaving 56 products. The commutative property can be used to help students remember half of these products once the other half has been learned. For example, if a student knows 7 imes 8, the student then knows that 8 imes 7 is the same. The remaining 28 products can be connected to known products by using the associative or distributive properties. For example, 6 imes 7 can be thought of as (6 imes 6) + 6, an application of the distributive property of multiplication over addition, and 5 imes 4 can be thought of as $(5 \times 2) \times 2$, an application of the associative property of multiplication.

Modeling Division

Division of whole numbers may be modeled by separating a set of objects into equivalent subsets and asking appropriate questions. Division may also be represented with continuous models such as lengths and areas, but we focus primarily on the set models.

Finding-How-Many-in-Each-Subset Model. One model for division involves separating a set into a known number of equivalent subsets and asking the question, "How many are in each subset?" The following real-world situation illustrates this model. Suppose that the envelopes containing the 300 invitations to a club function need to be bundled for distribution and that 25 large rubber bands are available to wrap around the bundles. When you bundle the set of 300 envelopes into 25 bundles and ask the question, "How many invitations are in each bundle?" you are demonstrating a fundamental meaning of division, as illustrated in Figure 2.26.

If you think of starting with the 300 envelopes and 25 rubber bands, you could share envelopes with the rubber bands by placing one envelope in each rubber band in turn until all the envelopes are distributed. Then you could count the number held by each rubber band. This approach suggests a sharing, or partitioning, model of division.

Finding-How-Many-Subsets Model. A second model for division involves separating a set into equivalent subsets of known size and asking the question, "How many subsets?" Figure 2.27 illustrates this model. To understand Figure 2.27, suppose that 300 invitations to a club function have been placed in envelopes

A total of 300 envelopes, with 25 bundles. How many in each bundle?

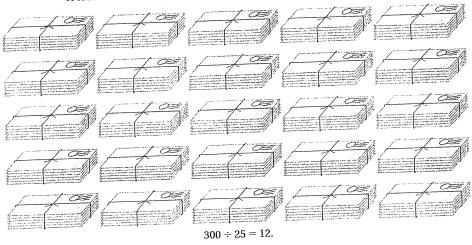


FIGURE 2.26
Partitioning model: Division as separating and finding how many in each subset.

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re es A total of 300 envelopes, with 25 in each bundle. How many bundles?

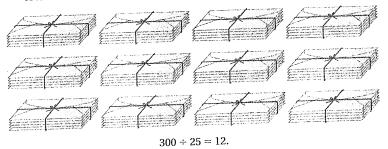


FIGURE 2.27

Measurement model: Division as separating and finding how many subsets.

that must be bundled for distribution. A stack of 25 envelopes seems to be convenient for bundling (and easy to count when combining bundles). When you bundle the set of 300 envelopes with 25 in each bundle and ask the question, "How many bundles are there?" you are demonstrating a fundamental meaning of division.

If you think about removing bundles of 25 to put in a mailbag, you could ask the question, "How many bundles (subsets) of 25 can be taken away from 300?" This question suggests a **repeated subtraction**, or **measurement**, **model of division**. You could think about taking away bundles (units) of 25 envelopes at a time and write $300 - 25 - 25 - 25 - \cdots = 0$. By counting the number of 25s subtracted to end up with 0 envelopes, you find the number of bundles that can be made and at the same time show that $300 \div 25 = 12$.

Mini-Investigation 2.8 is intended to help you extend your understanding of these two models for division of whole numbers.

Talk about which of the division models represents a process of one for Jim, one for Jan, one for Sue, and so on, that you might use to share some items with some friends.

MINIFINVESTIEVATIONE

Solving a Problem

What two problems can you state that can be solved by using the division 24 $\,\div\,$ 6 and that illustrate the two models of division described in Figures 2.26 and 2.27?

Example 2.13 provides a brief look at how continuous models rather than discrete set models might be used.

Example



Problem Solving: Art Supplies

Solve the problems in parts (a) and (b) and indicate which problem is a continuous counterpart of the finding-how-many-subsets model and which is a continuous counterpart of the finding-how-many-in-each-subset model:

- a. Mrs. Chance has a roll of butcher paper 50 feet long. She wants to cut it into 2-foot lengths for her students to use in art class. Will she have enough for her class of 22 students?
- b. Mrs. Chance has a roll of butcher paper 50 feet long. Only 10 students want to work with butcher paper today. If she wants to distribute the entire length of paper equally to the 10 students, how much paper should she give each of them?

SOLUTION

- Mrs. Chance wants to find the number of 2-foot pieces of paper in a 50-foot roll, so she divides: $50 \div 2 = 25$, or enough paper for 25 students. Because she knows the total length and the length of one piece and is trying to find the number of pieces, this problem is similar to finding how many subsets.
- Mrs. Chance has 50 feet of paper and wants to find the length of a piece that she can cut for each of 10 students, so she divides: $50 \div 10 = 5$, or 5 feet of paper for each student. Because she is trying to find the length of each piece, this problem is similar to finding how many in each subset.

YOUR TURN

Practice: Solve problems (a) and (b), and tell which model of division applies to each solution:

- a. Rikki has 36 feet of ribbon to use in wrapping presents. If each present takes 4 feet, how many presents can she wrap with the ribbon?
- b. Rikki has 36 feet of ribbon to use in wrapping presents. If she has four presents to wrap, what is the maximum amount of ribbon she could use for each present?

Reflect: In general, what question does division answer when you know the total length of a strip and are given the following information?

- a. the length of each piece
- b. the number of equal pieces to be cut

Using Multiplication to Define Division

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Division as the Inverse of Multiplication. The following real-world situation helps explain the special relationship between multiplication and division. How many tables that seat 8 people will you need to provide banquet seating for 72 people? You might naturally first think about the number sentence ? \times 8 = 72. However, to find the missing number of tables, you might actually complete the division $72 \div 8 =$? Note that the answer to the division in the second number sentence is the missing factor in the first number sentence. This situation illustrates the idea that multiplication, represented by putting several equivalent sets together, and division, represented by separating a set into several equivalent subsets, are inverse operations. That is, in doing multiplication, you multiply two factors to find a product, whereas in doing division, you divide the product by one factor to get the other factor:

Factor Factor Product
$$9 \times 8 = 72$$

Product Factor Factor
 $72 \div 8 = 9$

So the answer to a division sentence is one of the factors in the related multiplication sentence. This outcome suggests the following definition:

In the **division of whole numbers** a and b, $b \ne 0$, $a \div b = c$ if and only if c is a unique whole number such that $c \times b = a$. In the equation, $a \div b = c$, a is the **dividend**, b is the **divisor**, and c is the **quotient**. The operation $a \div b$ may also be written as $\frac{a}{b}$ or as $b)\overline{a}$.

Division as Finding the Missing Factor. The inverse relationship between multiplication and division is the basis for a slightly different view of division: finding the missing factor. That is, in multiplication you know two factors and find the product. In division, you know the product and one factor and find the other factor:

Product Factor Factor
$$36 \div 3 = ?$$

When asked to find the quotient $36 \div 3 = ?$, think of 36 as the product and 3 as one of the factors. Then ask, "What factor multiplied by 3 gives the product 36?"

Why Division by 0 Is Undefined. The view of division as finding the missing factor helps you deal with the division when 0 is involved. Equations such as $8 \div 0 = ?$ have no whole number that when multiplied by 0 gives 8 or any number other than 0. For the equation $0 \div 0 = ?$ every number multiplied by 0 gives 0. Because equations such as $8 \div 0 = ?$ have no solution and the equation $0 \div 0 = ?$ has infinitely many solutions, division by 0 is said to be undefined. In other words, the definition for division cannot be applied to a divisor of 0. However, dividing 0 by another number, say, $0 \div 8$, gives the quotient 0.

Comparing Division to Multiplication

Because the operation of division is defined in terms of multiplication, a natural question is whether the properties for multiplication of whole numbers hold for division. We examine this question in Example 2.14 by looking for counterexamples for each of the properties of multiplication as they relate to division of whole numbers.

Example 7

Comparing Multiplication and Division Properties

Does whole-number division have the closure property?

SOLUTION

As with multiplication, division of whole numbers would have the closure property provided that when we divide two whole numbers, the quotient is a unique whole number. As $11 \div 3$ doesn't have a whole-number answer, it serves as a counterexample proving that division is *not* closed for the set of whole numbers.

YOUR TURN

Practice: Provide counterexamples showing that the commutative, associative, and distributive properties for multiplication of whole numbers don't hold for division.

Reflect: Why is 1 *not* the identity element for whole-number division?

The counterexamples presented in Example 2.14 demonstrate that division doesn't have the same properties as multiplication.

Now that we have explored the operations of addition, subtraction, multiplication, and division, we need to take a brief look at how they relate to one another. As Figure 2.28 shows, the inverse relationship between addition and subtraction parallels the inverse relationship between multiplication and division. Also, the repeated addition relationship between addition and multiplication parallels the repeated subtraction relationship between subtraction and division.

The Division Algorithm for Whole Numbers

Although the mathematical definition for division of whole numbers restricts its use to numbers that "come out even," as in $24 \div 4 = 6$, whole-number division is often used to model physical situations that involve making groups of equal size whereby part of a group is "left over." For example, if you are setting up a conference room for 25 people and want to have 4 people at each table, you might find $25 \div 4$ and note that you could have six full tables with 1 person left over. In such situations when we want to describe $a \div b$ ($b \ne 0$) and no whole number c exists such that

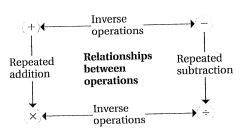


FIGURE 2.28

Relationships between operations.

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 $c \times b = a$, we use the **division algorithm.** Essentially, it extends the definition of division to include the possibility of a **remainder**.

The Division Algorithm

For any two whole numbers a and b, $b \neq 0$, a division process for $a \div b$ can be used to find unique whole numbers q (quotient) and r (remainder) such that a = bq + r and $0 \leq r < b$. For a = 25, b = 4, q = 6, and r = 1, $25 = (4 \times 6) + 1$.

In vertical form, we write

$$\begin{array}{r}
6 \text{ remainder 1} \\
4)25 \\
\underline{-24} (4 \times 6) \\
1
\end{array}$$

We can also use a calculator with integer division capability to illustrate the division algorithm. For example, using integer division to express $25 \div 4$ may result in the following screen:

In the symbolism of the division algorithm, a=25 and b=4. The calculator produces the quotient, Q=6, and the remainder, R=1. From the calculator display, we can conclude that $25=(4\times 6)+1$. Based on this example, you should be able to predict what will be displayed on the following screen when you press ENTER and to express the data in a form described in the division algorithm:

Mini-Investigation 2.9 extends the meaning of the division algorithm.

Talk about how the idea of the division algorithm would change if these words were omitted from the definition.

MINIFINVESTIGATION

Using Mathematical Reasoning

Why is the phrase "and $0 \le r < b$ " included in the definition of the division algorithm for whole numbers?



For children, learning about and using operations involves more than just memorizing basic facts and properties. Using an operation also requires practical, common-sense reasoning. For example, if 27 students are going on a field trip and each car will hold 4 students, how many cars will be needed for the field trip? This situation is one of forming equivalent subsets, so the problem can be solved by using ideas from the division algorithm: 27 \div 4 = 6 full cars and 3 students left over because $(4 \times 6) + 3 = 27$ and 0 < 3 < 4. Students need to be able to interpret the results of the division algorithm in terms of the problem being solved. In this case, the answer to the problem is that seven cars will be needed for everyone to be able to go on the field trip.

Problems and Exercises for Section 2.3

A. Reinforcing Concepts and Practicing Skills

1. Use sets to verify the following products:

b. $5 \times 10 = 50$ **a.** $8 \times 4 = 32$

c. $6 \times 0 = 0$ Show the union of 6 empty sets.

2. Use lengths on a number line to verify the following products:

a. $8 \times 4 = 32$

b. $5 \times 10 = 50$

c. $6 \times 0 = 0$

3. Use a rectangular array to verify each of the following products:

a. $8 \times 4 = 32$

b. $5 \times 10 = 50$

c. $6 \times 0 = 0$ No array, or rectangle, of 0 length

- 4. Create a word problem involving joining equivalent sets to go with the equation $4 \times 30 = 120$.
- 5. Create a word problem involving joining segments of equal length to go with the equation $4 \times 30 = 120$.
- 6. Write a multiplication equation and find the area of each of the following rectangles:
 - a. a rectangle 12 units high and 6 units wide
 - b. a rectangle 42 units high and 1 unit wide
 - c. a rectangle $\frac{b}{2}$ units high and 4b units wide

Write a multiplication equation for and find the answer to each of the following questions in Exercises 7-9.

- 7. If a furniture manufacturer has four styles of chairs and six different upholstery fabric choices for each chair, how many different chairs can be made?
- 8. A gift basket store sells three kinds of chocolate, five kinds of cheese, and three kinds of gourmet coffee. If the store has a special on a basket that contains one selection each of chocolate, cheese, and coffee, how many different basket specials can be made?
- 9. If a department store wants to advertise 64 different gift wrappings (based on a choice of one type of paper and one color of bow), how many different types of

wrapping paper and colors of bows should it plan to have on hand?

10. For the equation, $25 \times 4 = 100$, create an accompanying word problem involving

a. a Cartesian product. See Exercises 7-9 for examples.

b. an array.

c. the area of a rectangle.

11. Which properties verify the following conjectures? Model each situation with an appropriate equation or mathematical description.

a. To remember 8 \times 4, I just need to remember 4 \times 8. *

b. It's easy to multiply with 0. It doesn't even matter what the other factor is. Zero property of multiplication

c. I can never remember what 8×6 is. But I can remember that 8×5 is 40. So I mentally add on one more 8 to get 48.

d. I almost made a mistake on my worksheet because in one place I put that 9×3 is 27 and in another place I put that it is 29. But I know that it has the same answer every time, so I checked them both to see which one was right.

e. I like to multiply by 10 or 100. The digits just move to different place-value positions.

12. Use the distributive property of multiplication over addition to find each product:

b. $4 \times (5 + 6)$ 44 **a.** $(20 + 5) \times 3$ 75

c. $(x + 10) \times (3x + 2)$ 3x² + 32x + 20

For the equation $24 \div 6 = 4$, create an accompanying word problem for each situation in Exercises 13-17.

13. Separating a set into same size equivalent sets

14. Separating a segment into same-length segments

15. Separating a set into a specific number of equivalent sets *

16. Separating a segment into a specific number of segments of equal length

17. Finding the missing factor

18. Use the t a. 1 c. 1

19. Use divis ble, mak a. I

c.) **20.** Use equa tipli a. 1 **c.** *T*

21. Use quot a. 2 **b.** 2 c. 4

B. Dec 22. Stuc

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d. Y

18. Use a calculator and repeated subtraction to find each of the following quotients:

a. 143 ÷ 11

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b. 108 ÷ 9 🐠

c. 1,404 ÷ 52 Subtract 52 twenty-seven times

19. Use the definition of division to rewrite the following division equations as multiplication equations. If possible, find a whole-number value for each variable to make the statements true.

a. $18 \div 6 = n$

b. $25 \div x = 5$

c. $y \div 42 = 126$

 $\mathbf{d.} \ \ 0 \div b = c \quad \blacksquare$

20. Use the definition of division to determine the division equations that are related to each of the following multiplication equations:

a. $15 \times 3 = n$

b. $9 \times y = 9$

c. $r \times s = t$

d. $8 \times 0 = 0$

21. Use the division algorithm for whole numbers to find each quotient and remainder, and then justify your answers:

a. $26 \div 3$ 8 R 2; 8(3) + 2 = 26

b. $292 \div 21$ 13 R 19; 21(13) + 19 = 292

c. $4 \div 7 + 0 + 4 \div 0 = 4$

- B. Deepening Understanding
- 22. Study the following diagram:

Addition



- i. Characteristics of addition that are different from multiplication
- ii. Characteristics that are common to both operations
- iii. Characteristics of multiplication that are different from addition
- a. Complete the diagram to illustrate how the characteristics of multiplication and addition of whole numbers compare.
- **b.** Prepare a similar diagram for subtraction and division.
- 23. Predict the number of ordered pairs that will be generated from the Cartesian products in (a)–(c). Then find the Cartesian products and compare the number of ordered pairs in each one to your predictions. [Hint: Make a three-column (predicted n(C), C, and actual n(C)), three-row (a, b, c) table and record your answers.]

a. $\{1, 2, 3, 4\} \times \{\} = C$

b. $\{r, s, t\} \times \{a, b\} = C$

c. $\{a\} \times \{5, 6, 7, 8, 9\} = C$

d. Write the multiplication equation related to each of the Cartesian products.

- 24. Using the union of two sets to define addition requires that the sets be disjoint. Is that condition necessary for the two sets in the Cartesian product definition for multiplication? Why or why not? Give an example to support your response.
- 25. What role does multiplication play in our base-ten numeration system?
- **26.** The product 12×6 can be found by thinking, $(10 + 2) \times 6 = (10 \times 6) + (2 \times 6) = 60 + 12 = 72$.
 - a. What properties of multiplication are being used?
 - **b.** Use these properties to verify the product 12×64 .
- 27. Use the distributive property of multiplication over addition to rewrite each of the following sums as the product of two factors, where one of the factors is a sum:

a. 4rs + 16r 4r(s + 4)

b. $27a^2 + 81a + 9 \ 9(3a^2 + 9a + 4)$

c. 24a + 14b + 20c = 2(12a + 7b + 10c)

28. The basic facts table for multiplication of whole numbers is as follows:

×	0	1		3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

- a. Describe how the closure property of multiplication is reflected in the table.
- **b.** Describe how the multiplicative identity property is reflected in the table.
- c. Describe how the commutative property of multiplication is reflected in the table.
- d. Describe how the zero property of multiplication is reflected in the table.
- e. Describe how the distributive property of multiplication over addition can be illustrated by the table.
- f. How are these properties important in learning basic multiplication facts? Answers vary.
- 29. How can the multiplication table be used to solve division equations?
- **30.** Division can be used to model both separating a quantity into groups of a given size and separating a quantity into a given number of groups of equal size.
 - a. How are these situations different? (For example, what different questions are asked? What different actions are represented?)
 - b. How are these situations alike?

- 31. Consider the following statement: $(q \times 12) + r = 64$.
 - a. Identify all possible pairs of whole numbers for q and r that make the sentence true.
 - b. Which pairs of values for q and r satisfy the division algorithm for whole numbers? q = 5 and r = 4
 - c. What could happen if the division algorithm didn't include the restriction that r is greater than or equal to 0 and less than the divisor?
- 32. The distributive property of division over addition has the form

For whole numbers a, b, and c, with $c \neq 0$, $(a + b) \div c = (a \div c) + (b \div c)$, provided that whole number quotients exist for each division expression.

- a. Identify four sets of three whole numbers that illustrate this distributive property of division over addition.
- b. What patterns do you see in these sets of whole numbers?
- c. Why does the distributive property of division over addition work only in particular formats?
- C. Reasoning and Problem Solving
- 33. Describe as many patterns as you can find in the multiplication facts table in Exercise 28. Use the commutative, associative, and distributive properties to explain why these patterns occur. Answers vary.
- **34.** Consider the data shown in the following picture graph and frequency chart. Answers vary.

Picture Graph

	Picture	Grapu	
	Favorite leis	ure pastime	Ž.,
Reading	Crafts	Sports	Audiovisual recreation ?
	69		

= 5 students.

Frequency Chart

~ ~	-1J		
Reading	Crafts	Sports	Audiovisual recreation
ş.	?	?	33
		- 0	Reading Crafts Sports ? ? ?

- a. Write two multiplication word problems and two division word problems reflecting the relationship between the data in the graph and the chart.
- b. Write an equation that represents the solution to each of your problems. Could either solution be represented by more than one equation?

- 35. The Candy Dish Problem. Eleanor made a deal with her mom. When she opened a bag of candy to put in her mom's four candy dishes, she would put an equal amount in each dish and then get to eat the ones left over. Last year, her mother received three new candy dishes as presents, and Eleanor believes that she has been getting to eat more leftover candy this year. Could that be true? Explain why or why not.
- **36.** Consider an equation that involves both multiplication and division of whole numbers: Augment vary.

$$a \times b \div c = q$$
 (with remainder $0 \le r < c$).

- a. Taking into consideration the use of the division algorithm to allow a remainder, make predictions about what happens to q and r when a increases, when a decreases, when a increases and a decreases, and so on.
- b. Test your predictions using a spreadsheet or graphing calculator.
 - **c.** State your conclusions in statements such as, If *a* and *b* both increase, then *q*
 - **d.** Arrange your conclusions into two categories: q decreases if . . .; and q increases if . . .
 - e. Share your categories with other students and look for patterns.
- 37. Which of the following sets of whole numbers has (have) the closure property for multiplication?
 - a. {1, 3, 6}
 - **b.** {0,1}
 - **c.** {10, 20, 30, 40, ...}
 - **d.** {0}
 - e. the set of even whole numbers
 - f. the set of odd whole numbers
- 38. True or false? No finite subset of whole numbers has the closure property for multiplication. Support your answer.
- **39.** Why can't 0 be used as a divisor? Use the definition of division to explain why the following statements are true:
 - **a.** $0 \div a = 0$.
 - **b.** $a \div 0$ is undefined.
 - c. $0 \div 0$ is undefined.
- 40. Return to the picture you drew in Mini-Investigation 2.7 to show the similarities and differences between multiplication and division. What changes would you make to this picture now? Why? Answers vary.
- D. Communicating and Connecting Ideas
- 41. The traditional sequence for teaching operations in the early grades is addition first, followed by subtraction, multiplication, and division. Some early arithmetic texts addressed the four operations in the order of addition, multiplication, subtraction, and division.

With a team or oral form approach.

42. Historical proposed the plication are multiplication two numbers which shall other hath (1953), p.

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- 42. Historical Pathways. Various definitions have been proposed throughout history for whole-number multiplication and division. An early (1677) definition of multiplication stated, "Multiplication is performed by two numbers of like kind, for the production of a third, which shall have such reason [ratio] to the one, as the other hath to unite [unity, or the number 1]" [Smith (1953), p. 103]. This definition could be restated as
- $a \times b = c$ means that $c \div a = b \div 1$. Compare this definition to the definitions of multiplication used in this section.
- 43. Making Connections. For many pairs of whole numbers, a and b, no whole number represents $a \div b$. What types of numbers are needed to bring the closure property to division? What physical situations could these numbers model? How would these numbers be related to the set of whole numbers? How could these numbers be helpful in using division to compare two whole numbers?

Section

Numeration



Numeration Systems The Hindu-Arabic Place-Value
Numeration System Other Early Numeration
Systems Comparing Numeration Systems

In this section, we examine symbols used to represent whole numbers, with an emphasis on clarity and efficiency of communication. We consider some early methods of representing whole numbers and compare them with the system for representing whole numbers that is used today.

Essential Understandings for Section 2.4

For any number, the place of a digit tells how many ones, tens, hundreds, and so forth are represented by that digit. A digit times its place value gives the value of the digit. You can add the value of the digits together to get the value of the number.

Each place value to the left of another is 10 times greater than the one to the right.

The structure of the base-ten numeration system produces many numerical patterns.

Place-value periods (thousands, millions, billions, etc.) are used to read and write numbers.

Whole numbers can be named in equivalent ways using place value.

Whole numbers can be compared by analyzing corresponding place values.

Mini-Investigation 2.10 may help broaden your ability to communicate number ideas.

Write a paragraph evaluating the strengths and weaknesses of each of your methods.



Communicating

Without using the number symbols 0–9 or the number words zero, one, two, . . . nine, devise three different ways to communicate to someone else the number of people in your class.

Envision sets.	Think about a number.	Write a numeral.
	TW0	2

FIGURE 2.29

Making connections between objects, whole-number ideas, and numerals.

Numeration Systems

In Section 2.1, we used sets to develop an understanding of whole numbers and used the symbols 0, 1, 2, 3, and so on, to represent these numbers. To understand how these symbols are used to communicate whole numbers, recall that a symbol is different from the number it represents. Just as the word *cat* is not itself an animal, the symbol 2 is not itself a number.

A written symbol, such as 2, that represents a number is called a **numeral**. Sometimes a numeral is referred to as a *name for a number*. For example, some familiar numerals (or names) for the number two are 2 and II. Figure 2.29 illustrates the connection between sets of two objects, the whole-number idea two, and the numeral 2.

Using numerals to communicate number ideas requires a method for representing all the whole numbers in some systematic way. That is, some type of *numeration system* is needed.

A **numeration system** is an accepted collection of properties and symbols that enables people to systematically write numerals to represent numbers.

The Hindu-Arabic Place-Value Numeration System

The **Hindu–Arabic numeration system**, developed by ancient Indian and Arabic cultures, is still in widespread use today and probably is the most familiar example of a numeration system. The ideas of grouping by tens and *place value* provide the cornerstones of the Hindu–Arabic numeration system.

In a numeration system with **place value**, the *position* of a symbol in a numeral determines that symbol's *value* in that particular numeral. For example, in the Hindu–Arabic numeral 220, the first 2 represents two hundred and the second 2 represents twenty.

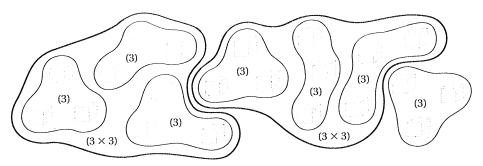


FIGURE 2.30 Grouping by threes: 2 groups of 3×3 , 1 group of 3, and 0 leftovers = 210 three.

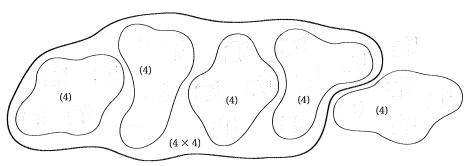


FIGURE 2.31
Grouping by fours:1 group of 4×4 , 1 group of 4, and 1 leftover = 111_{four} .

A look at grouping schemes other than grouping by tens can help you better understand the Hindu–Arabic system. For example, if the grouping is based on threes, a set of three beans makes a group of the first size (3), a set of three of these groups makes a group of the next larger size (3 × 3 or 3²), three of these groups make a group of the next larger size (3 × 3 × 3 or 3³), and so on. In Figure 2.30, a set of 21 cubes has been grouped by threes. The same set of cubes, grouped by fours, is shown in Figure 2.31. The sizes and numbers of groups and leftovers change with each different grouping, but the number of cubes and the grouping procedure remain the same. The pattern used for writing and interpreting place-value numerals is based on this grouping procedure. The number of groups of each size is recorded in the numeral, with the rightmost digit representing the leftovers and the digits to the left representing increasingly larger groups.

To know what quantity is being represented by a place-value numeral, you must know the sizes of the groups on which the numeral is based. The group size used determines the **base** of the numeration system. Using groups of ten is the foundation of the Hindu–Arabic numeration system, and for this reason the system is often called the *base-ten place-value numeration system*. Since grouping by tens is the basis of the numeration system ordinarily used, only numerals in bases other than ten are written with a word indicating the base. For example, 120 indicates a base-ten numeral; $120_{\rm five}$ indicates the base-five numeral. Example 2.15 helps clarify the role of different bases in a numeration system.

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Example

Expressing Numerals with Different Bases

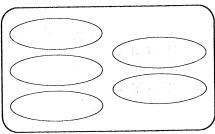
Explain why the quantity of tiles shown can be expressed as (a) 27 in base ten and (b) 102 in base five:

SOLUTION

a. Using a base-ten system, we can group these tiles into two groups of ten with 7 tiles left over:

The numeral 27 represents the two groups of ten and the 7 ones.

b. Using a base-five system, we can group these tiles into groups of 5 and have enough of these groups of 5 to make one larger group of 5 fives, with 2 tiles left over:



No group of 5 is left over, so we need to use a 0 in that position in the numeral. We record this quantity as $102_{\rm five}$ so that everyone knows the group sizes used.

YOUR TURN

Practice: Draw pictures and explain why the number of tiles shown can be represented by both 38_{ten} and 123_{five} .

Reflect: What does the digit 1 in the numeral 123_{five} represent?

We can see some interesting patterns in place-value numeration systems of different bases by arranging consecutive numerals into square grids. Figure 2.32 shows a three-by-three grid for base-three numerals, a five-by-five grid for base-five numerals, and a ten-by-ten grid for base-ten numerals. The base-ten grid is often

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1 2 10	1	2	3	4	10
11 12 20	11	12	13	14	20
21 22 100	21	22	23	24	30
Base-Three Grid	31	32	33	34	40
buoo IIII oo o	41	42	43	44	100

Base-Five Grid

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Base-Ten Hundred Chart

FIGURE 2.32

Numeral patterns in place-value numeration systems

called a hundred chart. Note that 2 and 10 are consecutive numbers in base three because 2+1=3 and $10_{\rm three}=3_{\rm ten}$.

Table 2.1 summarizes some other important patterns in a place-value system and generalizes the patterns for every base, $b \ge 2$. Using the ideas in Table 2.1, we can show base-ten interpretations of numerals expressed in different bases:

$$1324_{\text{five}} = 1(125) = 3(25) + 2(5) + 4 = 229;$$

$$1324_{\text{eight}} = 1(512) + 3(64) + 2(8) + 4 = 724;$$

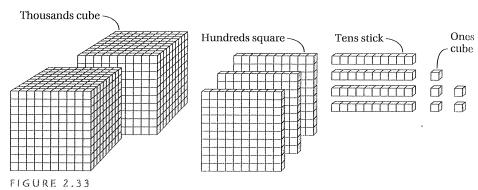
$$1324_{\text{ten}} = 1(1000) + 3(100) + 2(10) + 4 = 1324;$$

$$1324_b = 1(b \times b \times b) + 3(b \times b) + 2(b) + 4.$$

Not only do numeration systems with different bases have applications in computer systems, they also play an important role in helping you fully understand the characteristics of the base-ten numeration system. When you try to write a numeral

TABLE 2.1 Place Values for Different Bases in the Numeration System

Base	Place Valu	ies for a 4-Digit Nume	ral	
Two	two × two × two	two × two	two	one
Five	five \times five \times five	$five \times five$	five	one
Eight	eight × eight × eight	eight × eight	eight	one
Ten	ten X ten X ten	ten × ten	ten	one
:				
b	$b \times b \times b$	$b \times b$	b	one



Base-ten block model representing the numeral 2,345.

in a different base, you're forced to think about the importance and use of both grouping and place value.

Models of Base-Ten Place Value. The fundamental idea of place value in the base-ten numeration system can be clarified with physical models. Different types of models enhance understanding at different levels of abstraction. **Proportional models** for place value actually exhibit the proportional differences in the values of the digits in a numeral.

Base-ten blocks are an example of a proportional model for place value. A set of base-ten blocks contains unit cubes (usually 1 cubic centimeter in size), a tens stick (10 unit cubes put together), a hundreds square (10 tens sticks put together), a thousands cube (10 hundreds squares put together), and so on. Base-ten blocks are used in Figure 2.33 to model a 4-digit numeral.

In **nonproportional models** of place value, the actual quantity expressed by the numeral isn't visible—usually just being represented by some object or set of objects. In a nonproportional colored-chip model, a red chip might represent one, a blue chip could represent ten ones or ten, a yellow chip might represent ten tens or one hundred, and a green chip might represent ten hundreds or one thousand. Based on these categories, the arrangement of chips shown in Figure 2.34 represents the number 3,462.

Using Expanded Notation. When a numeral is written to show the sum of its digits times the value of each place, we say the numeral is written in **expanded notation**. For example, we can write the base-ten numeral 5,283 in expanded notation as

$$5,283 = 5(10 \times 10 \times 10) + 2(10 \times 10) + 8(10) + 3$$

or

$$5,283 = 5(1000) + 2(100) + 8(10) + 3.$$

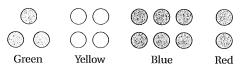


FIGURE 2.34

Colored-chip model representing the number 3,462.

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Elementary school students need to have a deep understanding of place value in order to apply it in comparing numbers and adding, subtracting, multiplying, and dividing numbers. Below are three students' responses to the following question:

How are the 5s in 585 different?

one Five is in the humbreds place.
One five is in the ones place.

The 5 is lower than 585 and 585 is longger than 5

Five in the cno Place means 5. 5 in the hundreds Place means 500

Source: © 2006 Heinemann Publishing. www.heinemann.com/math, Question_ID = 198, # 1,2,3.

Which student do you think exhibited the deepest understanding of place value? What follow-up questions might you ask each student to find out more about his or her understanding of place value?

If we use exponents to show how many times 10 is used as a factor, the use of expanded notation becomes even more efficient:

$$5,283 = (5 \times 10^3) + (2 \times 10^2) + (8 \times 10^1) + (3 \times 1).$$

Example 2.16 further illustrates the use of expanded notation.

Example



Expressing Numbers Using Expanded Notation

Express 2,150,389 in expanded notation.

SOLUTION

Meei-Ling's thinking: According to the patterns in place value,

$$2,150,389 = (9 \times 1) + (8 \times 10) + (3 \times 100) + (5 \times 10,000) + (1 \times 100,000) + (2 \times 1,000,000).$$

Ex

Michael's thinking: Exponents indicate how far a digit is from the ones place, so in expanded notation this number is

$$(2 \times 10^6) + (1 \times 10^5) + (5 \times 10^4) + (0 \times 10^3) + (3 \times 10^2) + (8 \times 10^1) + (9 \times 1).$$

YOUR TURN

Practice: Use exponents to express 48,042 in expanded notation.

Reflect: How would you change the expanded notation for 48,042 to express a number that is 100 less? 10 less? 110 less? \blacksquare

Note that when you use exponents to express numbers in expanded notation, the thousands position is represented by the exponent 3, the hundreds by the exponent 2, and the tens by the exponent 1. Continuing this pattern leads to the conclusion that the ones position should be represented by the exponent 0. Using this notation means that 4,378, for example, could be represented as

$$(4 \times 10^3) + (3 \times 10^2) + (7 \times 10^1) + (8 \times 10^0).$$

Using Periods to Represent Large Numbers. The pattern that relates the value of a place in a numeral to the value of the place to its left allows the writing of numerals to express numbers far greater than those that can be shown with simple objects or drawings. To do so, we use successive groups of three digits, called **periods**. As illustrated in Figure 2.35, the three digits in each period are the number of

Length of Life

How old is someone who has lived 1 million hours? Estimate your answer by choosing one of (a)-(e).

- a. less than 1 year old
- b. between 1 and 10 years old
- c. between 10 and 50 years old
- d. between 50 and 100 years old
- e. more than 100 years old Determine a way to check your estimate.

Hundred Ten One Hundred Ten One Hundred Ten One Billions Millions Thousands Ones

456, 703, 995, 652

Read "456 billion, 703 million, 995 thousand, 652."

FIGURE 2.35

Representing large numbers with numerals and periods.

hundreds, tens, and ones that indicate how many of the designated value of a period—ones, thousands, millions, billions, and so on—are involved. Some additional periods, to the left of billions, are trillions, quadrillions, quintillions, and so on.

Using Place Value to Compare Numbers. In Section 2.1, we compared two whole numbers by looking at sets for the numbers and matching elements of the sets to see which set had elements left over. In Section 2.2, we used addition to compare two whole numbers. By combining these set-matching and addition procedures, we can develop a procedure for comparing two numbers using the place-value relationships of the digits in the numerals. Example 2.17 illustrates the use of place-value ideas in comparing two numbers.

Example



Using Place Value to Compare Numbers

Which is greater, 4,800 or 4,756? Explain your reasoning.

SOLUTION

Angela's thinking: If I made the numbers with base-ten blocks, each one would take 4 thousands cubes. But 4,800 would need 8 hundreds squares, and 4,756 would only need 7. The larger number of tens and ones pieces in 4,756 wouldn't be enough to make up for the extra hundreds piece. So 4,800 is greater than 4,756.

Rodney's thinking: I lined the digits of the two numbers up according to their place-value positions. When I looked at the leftmost digits, I noticed that each number has the same number of thousands (4). Then I looked at the hundreds digits; 4,800 has 8 hundreds but 4,756 has only 7 hundreds. So 4,800 is greater than 4,756.

YOUR TURN

Practice: Which is smaller, 3,402 or 2,799?

Reflect: Can you choose digits for the blanks so that the number 2 _ _ _ will be larger than the number 3,000? Explain.

Other Early Numeration Systems

Token Systems and Tally Systems. The actual history of how human beings moved from concrete objects to symbols to represent numbers has been lost for many cultures, but archaeological evidence indicates that in the area near present-day Iran, from 8000–3000 B.C., symbols in the form of clay tokens shaped like cones, spheres, disks, cylinders, and pyramids were used to represent measures of grain, oil, metal, and other goods in the economy. Thus one of the earliest known systems of numeration may be described as a **token system**, in which symbolic objects, or tokens, were used to represent quantities of actual objects.

In this system, each token represented one measure of a particular commodity, and sets of tokens were combined to represent larger quantities. Toward the end of the period, tokens also began to represent groups of measures (for example, a sphere meant 10 measures of grain), and sets of tokens began to appear enclosed in clay envelopes for ease of transporting and for storage in archives as long-term records. Because the quantity of tokens inside a clay envelope couldn't be determined without

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One



FIGURE 2.36

Sketches of early tokens and clay envelopes imprinted to show tokens inside.

breaking it open, scribes began making imprints on the outside of the clay envelopes. A circular dent in the clay represented a spherical token, and a triangular dent, a cone-shaped token (Schmandt–Besserat, 1992). Figure 2.36 shows a sketch of these early tokens and an imprinted clay envelope.

The realization that the imprints made the tokens themselves unnecessary was an early step toward an actual written numeration system. A **tally system** is based on establishing a one-to-one correspondence between a single mark and a single object so that the marks represent the number of objects. Later, grouping was used, as in representing seven hides with ||| |||, to simplify numeration systems. Tally systems are still used sometimes to record one object at a time and later determine the total.

Egyptian Numeration System. The ancient Egyptians developed a type of tally system slightly more sophisticated than those based on strict one-to-one correspondence in about 3400 B.C. Found written on papyrus, wood, pieces of pottery, and stone, the Egyptian numeration system used picture symbols, called *bieroglyphics*, for 1, 10, 100, 1,000, and so on, as depicted in Figure 2.37.

Eg	yptian symbol	Corresponding whole number
	Reed	One
\bigcap	Heel bone	Ten
0	Coiled rope	One hundred
\sum_{i}	Bent reed	One thousand
P	Pointed finger	Ten thousand
\odot	Burbot fish	One hundred thousand
4	Astonished man	One million

FIGURE 2.37

Egyptian hieroglyphs for whole numbers.

These symbols were combined and repeated as necessary to record whole numbers, such as \cap\cap\llllllll for 37. Example 2.18 compares the Egyptian numeration system with the Hindu-Arabic system.

Example



Writing Egyptian and Hindu-Arabic Numerals

- a. Use Egyptian symbols to represent 5,642.
- b. Use Hindu-Arabic symbols to represent



\$50LUTION

b. 240,318

YOUR TURN

Practice:

- a. Use Egyptian symbols to represent 1,354,201.
- b. Use Hindu-Arabic symbols to represent

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Reflect: What advantages does the Hindu–Arabic numeration system have over the Egyptian system?

Babylonian Numeration System. In ancient Babylon, numbers were recorded on clay tablets, with a stylus or writing stick used to make imprints in the clay. The **Babylonian numeration system**, based on multiples of 60, utilized only two symbols. These symbols could be formed easily in the clay with a stylus but had multiple meanings:

One group of 60
One group of 60 sixties (60²)

Ten ones $(10 \cdot 1)$ Ten groups of 60 $(10 \cdot 60)$ Ten groups of 60 sixties $(10 \cdot 60^2)$

The meaning of each symbol was determined by the context or the position in which it appeared. For example, the meaning of the single symbol that could mean 1, 60, or 60 sixties (3,600) would be 60 rather than 1 if it appeared in a record of the number of eggs consumed in a 2-month period by a person who liked eggs.

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Combinations of the two symbols, with spacing or size of symbol delineating different meanings of the same symbol, were used to represent whole numbers, as in the following diagram:

12 602 (1 ten, 2 ones) (10 sixties, 2 ones)

With a grouping number as large as 60, it is difficult to see the place-value patterns. Figure 2.38 illustrates some of them.

Base-Ten Hindu–Arabic numeral	Babylonian numeral	
1	Y	
2	A A	
3	Y Y Y	
• : • • • • • • • • • • • • • • • • • •	:	
10	≺	
11	< ▼	
12	< Y Y	
•	:` :	
20	< <	
21	∴	
22	< < 11	
•	:	
• 59	<<<< !	7 7 7 7
60		
61		· · · · · · · · · · · · · · · · · · ·
62	V V	
•	: ' '	
· 70		
71	A \langle 1	
The second control of	∀	
72 :		
	▼ 	
119	A	7 7 7 7
120	Y Y	
121	V V	Y

FIGURE 2.38
Babylonian numerals.

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atterns.

Babylonian numeral	Corresponding Hindu–Arabic numeral		
V Y Y	64		
V Y Y	Could mean $1(60^2) + 4(60) = 3,840 \text{ or}$ $1(60^2) + 4 = 3,604$		
< < V	$10(60^2) + 10(60) + 1 = 3,6601$		
< <pre><</pre>	42(60) + 34 = 2,554		
Y . \{ Y	$1(60^2) + 0(60) + 1 = 3,601$		

FIGURE 2.39

Examples of Babylonian numerals.

Much later, the Babylonians learned to help clarify different meanings by filling a blank place with a symbol meaning "nothing in this place," as in the last line of Figure 2.39. Several different symbols were used over the years to represent this idea of "nothing."

Figure 2.39 illustrates more Babylonian numerals and the equivalent Hindu-Arabic numerals.

Example 2.19 provides an opportunity to interpret a Babylonian numeral.

Example



Interpreting Babylonian Numerals

Suppose that a Babylonian had used a stylus to make the following numeral to show how many units of land were in a large valley:

Write the Hindu-Arabic numeral that gives the same information.

ESCLUTION

The numeral represents $12(60^2) + 0(60) + 41$, or 43,241.

§YOUR TURN

Practice: Write the Hindu-Arabic numeral that gives the same information as the Babylonian numeral



Reflect: What advantages does the Hindu–Arabic numeration system have over the Babylonian system?

Roman Numeration System. The numeration system developed between 500 B.C. and A.D. 100 by the Romans is still used today in certain contexts—for example, to indicate year dates on buildings and in credits on films, to number pages in the preface of a book, and to identify Super Bowl games. The Roman numeration system uses seven basic symbols to represent whole-number groupings:

Symbol	Whole Number
I	One
V	Five
X	Ten
L	Fifty
C	One hundred
D	Five hundred
M	One thousand

These seven symbols are combined and repeated as necessary to form Roman numerals. For efficiency, however, one of the conventions of the Roman numeration system is that no more than three of any one symbol are used in a numeral. Thus the quantity *four* is represented as IV, or one less than five, and *nine* is IX, or 1 less than 10. Very large numbers are represented by using a bar over a numeral to represent multiples of 1,000, as in \overline{C} for 100,000. Example 2.20 illustrates the use of symbols in Roman numerals.

Example >

Interpreting Roman Numerals

Identify the numbers represented by the following Roman numerals:

- a. MCCXLVII
- b. CCCXXIV
- c. CMXCIX
- \mathbf{d} . $\overline{\mathrm{LVI}}$

SOLUTION

Think about the symbols in groups. Then

- a. M = 1,000, CC = 200, XL = 10 less than 50, or 40, and VII = 7, so MCCXLVII = 1,247
- **b.** CCC = 300, XX = 20, and IV = 4, so CCCXXIV = 324.
- c. CM = 900, XC = 90, and IX = 9, so CMXCIX = 999.
- **d.** LVI = 56, so \overline{LVI} = 56,000.

YOUR TURN

Practice: Write the Roman numerals for the numbers between XXXV and LV.

Reflect: Does any pattern appear in the numbers (such as IV and IX) that are symbolized by subtractive representation?

Mayan Numeration System. The Maya civilization developed a system of numeration around Λ.D. 300 that consisted of three symbols—a shell-like symbol for 0, a dot for 1, and a bar for 5. Figure 2.40 shows the Mayan numerals for the numbers 0 through 19.

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m Roman numeration l. Thus the l less than represent of symbols

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system of ke symbol als for the

	•	. • •	• • •	• • • •
Zero	one	two	three	four
	•	• 8		
five	six	seven	eight	nine
	•		• • •	
ten	eleven	twelve	thirteen	fourteen
		• •		
fifteen	sixteen	seventeen	eighteen	nineteen

FIGURE 2.40 Mayan numerals.

The Mayan numeration system also included the use of place value, with the place-value positions arranged vertically and increasing by factors of 20, except for position three, which increased by a factor of 18, as shown in Figure 2.41.

Therefore, the Mayan numerals for the numbers 20; 486; and 867,959 would be written as shown in Figure 2.42.

The justification for the value of the Mayan symbol for 867,959 is shown in Figure 2.43. It is conjectured that the use of 18 instead of 20 in the third position is due to the resulting value of 360—a useful number in the Maya's recording processes for days and times.

Position 5	$20 \times 18 \times (20)^2$	===	144,000
Position 4	$20 \times 18 \times 20$	=	7,200
Position 3	20×18	==	360
Position 2	(20) ¹	20.000	20
Position 1	$(20)^{0}$	==	1

FIGURE 2.41

The first five place-value positions of the Mayan numeration system.

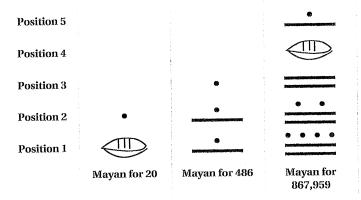


FIGURE 2.42
Example of Mayan numerals.

$20\times18\times(20)^2$	**************************************	******	$6 \times 144,000$	864,000
$20 \times 18 \times 20$		=	$0 \times 7,200$	0
20 × 18		=	10×360	3,600
(20)1		===	17 × 20	340
(20) ⁰		Wilder & Britanis	19 × 1	19
				867,959

FIGURE 2.43

Justification for Mayan numeral.

Example 2.21 uses this vertical place-value system to translate a Hindu–Arabic numeral into a Mayan numeral.

Example



Interpreting Mayan Numerals

Write the Mayan numeral that represents the same number as the Hindu-Arabic numeral 3,010.

SOLUTION

The Hindu–Arabic numeral 3,010 can be represented as $(8 \times 360) + 6(20) + 0(1)$. So, in the Mayan numeration system, the number would be written as



EYOUR TURN

Practice: Write the Mayan numeral that represents the same number as the Hindu–Arabic numeral 2,056.

Reflect: What advantages does the Mayan numeration system have over the Babylonian system?

Comparing Numeration Systems

The Hindu–Arabic system used today can be more clearly understood and appreciated by comparing and contrasting it with the other systems. In the following subsections, we compare systems based on grouping schemes, symbols, place value, and the use of zero. Using these criteria you will have an opportunity to judge the effectiveness of the different systems.

Grouping Scheme and Symbols. One way of comparing numeration systems is to look at the *grouping scheme* and *symbols* used. The Egyptian system grouped by tens, with a new symbol for ones, for tens, for ten tens, for ten tens, and so on.

Theoretically, the Egyptian numeration system required infinitely many symbols. In contrast, the Babylonian and Mayan systems had very few symbols. The Babylonian system at first had only two symbols and grouped by sixties. The two symbols, representing the whole numbers one and ten, were used repeatedly and in different positions to write larger numbers. The Mayan system used three symbols, representing zero, one, and five; grouped by twenties; and used the symbols repeatedly and in different positions to write larger numbers.

The Roman system involves a modified scheme of grouping by fives and a combination of basic symbols, with bars as needed to represent larger numbers. More and more new symbols would be needed to represent larger and larger numbers. The Hindu–Arabic system involves grouping by tens and uses only 10 symbols: 0, 1,

2, 3, 4, 5, 6, 7, 8, 9.

Use of Place Value. Another way to compare numeration systems is in terms of use of place value, where the *position* of a symbol in a numeral determines the value it represents. The use of place value allows the writing of large numbers by imaginatively arranging old symbols without having to invent new symbols. The Babylonians probably were the first people to reduce the number of symbols needed to represent quantities by using place value, but their system had a somewhat confusing type of place value, in which the meaning of the symbol was often open to interpretation. The Mayan system incorporated vertical place-value positions and did not have the ambiguity of the Babylonian system.

In contrast to the Babylonian and Mayan systems, the Egyptian system didn't utilize place value and required more and more symbols to represent larger and larger numbers. Like the Babylonian and Mayan systems, the Hindu–Arabic system is a place-value system, with each symbol having more than one possible value. For example, the symbol 2 can represent the whole numbers two or two hundred or two

thousand, depending on its position in a numeral. For example,

in 52, the 2 represents the whole number two;

in 25, the 2 represents the whole number twenty; and

in 256, the 2 represents the whole number two hundred.

The Hindu–Arabic system makes the most efficient use of place value, for—unlike the Egyptian, Babylonian, Roman, and Mayan systems—the Hindu–Arabic symbols are never used in the form of a tally. If a symbol is repeated in a numeral, each occurrence represents a distinct and different value, as in 2,222. Note that in the Roman system the location, or place, of a numeral makes a difference, such as the location of the I in IV and VI. However, a single I without a bar over it, for example, always has the value 1, regardless of its place in a numeral.

Use of Zero. Another criterion for comparing numeration systems is the use of **zero.** The Mayan system, the later Babylonian system, and the Hindu–Arabic system are the ones that have a symbol for zero. In these systems, the position of each digit in a numeral may be clearly indicated by the use of the symbol for zero when necessary. For example, in each of the following numerals, the use of zeros indicates clearly that the digit 2 is in the third position from the right and represents 200:

10,204; 5,280; and 200

Table 2.2 summarizes the grouping schemes, symbols, and use of place value and zero of the numeration systems discussed in this section. Mini-Investigation 2.11 provides an opportunity to analyze the numeration systems further.

-Arabic

-Arabic

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TABLE 2.2 Summary of Numeration System Characteristics

System	Grouping	Symbols	Place Value	Use of Zero
Egyptian	By tens	Infinitely many possibly needed	No	No
Babylonian	By sixties	Two	Yes	Not at first
Roman	Partially by fives	Infinitely many possibly needed	Position indicates when to add or subtract	No
Mayan	Mostly by twenties	Three	Yes, vertically	Yes
Hindu–Arabic	By tens	Ten	Yes	Yes

Write a paragraph supporting your conclusion about the firstand second-place winners.

MINIMARY CANON PARE Making a Connection

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If there were a "numeration system contest," which system in Table 2.2 do you think would win and which would come in second?

Problems and Exercises for Section 2.4

- A. Reinforcing Concepts and Practicing Skills
- 1. Indicate whether each of the following statements is about numbers or is about numerals:
 - a. If you cut vertically, half of 8 is 3. Numeral
 - b. If you double 2 you get 4. Manches
 - c. 5 is larger than 8. Name and
- 2. Write the number of objects as a
 - a. base-ten numeral.
 - b. base-two numeral.
 - c. base-five numeral.
- 3. a. Express the number 586 using base-ten blocks.
 - b. Express the number 4,392 using expanded notation without exponents.
 - c. Express the number 2,864,071 using expanded notation with exponents.
- 4. What quantity does the 3 represent in the following numerals?

 - **a.** 4352 Agrasque of 100 **b.** 324_{five} Agraps of 20
 - c. 311_b
- 5. Find the base-ten representation for each of the following numerals in other bases:
 - a. 344_{six}
- **b.** 202_{four} 34
- c. 110011_{two} 51
- 6. Find the representation of the number 256 in the following bases:
 - a. Base six
- **b.** Base twelve 194_{10 other}
- c. Base two 1000000000....

- 7. **a.** $42_{\text{seven}} = \underline{\qquad}_{\text{five}}$ five.
- 8. Use place value to order the following set of numbers from least to greatest: 962, 980, 983, 1000, 2023, 2245

2,222

- 2245 962 980 1,000 985
- 9. Describe the basic characteristics of a tally system.
- 10. Write the Hindu-Arabic numerals for the numbers represented by the following numerals from the Egyptian
 - a. ₹ 6 6 6 111
- 11. Write the Hindu-Arabic numerals for the numbers represented by the following numerals from the Babylonian system:
 - a. 🗸 🗸 🔻
 - b. **< < \ \ \ \ \ **
 - c. ****
- 12. Write the Hindu-Arabic numerals for the numbers represented by the following numerals from the Roman system:
 - a. MCCCIV BOX
- b. XLVIII
- **c.** CCCIX 369

- 13. Write the H resented by system:
- 14. Translate the 65 into the e
 - a. Egyptian b. Babylonia
 - c. Roman nu
 - d. Mayan nu
- 15. Describe the ing numeration
 - a. Egyptian:
 - b. Babylonia:
 - c. Roman an d. Mayan and
- 16. On a hundred the base-ten I
- 17. Use a hundred
 - a. 10 more th
 - **b.** 10 less that
 - c. 1 more tha
 - d. 1 less than
 - e. 11 more th
 - f. 11 less that
 - g. 9 more tha
- **h.** 9 less than 18. Twenty-five in
 - colored chips.) a. What is the
 - **b.** What is the
 - c. How and w numeration
- 19. Explain the nur single digits.
- 20. Use some or all numbers in bas sentences, and
 - a. When I dro way was mo
 - **b.** I spent abou \$____ more
 - c. I have ____ d. My roomma
 - e. The score of
 - f. I have _____ next month.

13. Write tl	ne H	indu	–Arabic nu	ımerals for	the n	umb	ers rep-
resented	l by	the	following	numerals	from	the	Mayan
system:	(21)	126,	(ii) 93, (c)	510			

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					€	•			
	•	1	KAR	actions.	电光压器			property contracts	NAME AND ADDRESS OF THE PERSONS ASSESSED.
ι.	THE RESIDENCE AND THE PARTY OF	b.	1000	(CAR 100 A)		especial S	c.	National and an arrangement of	SERVICION PART

- 14. Translate the Hindu–Arabic numerals 120, 59, 403, and 65 into the equivalent
 - a. Egyptian numerals.
 - b. Babylonian numerals.
 - c. Roman numerals.
 - d. Mayan numerals.
- 15. Describe the similarities and differences in the following numeration systems:
 - a. Egyptian and Hindu-Arabic
 - b. Babylonian and Hindu-Arabic
 - c. Roman and Hindu-Arabic
 - d. Mayan and Hindu-Arabic
- 16. On a hundred chart, what patterns are direct results of the base-ten numeration system?
- 17. Use a hundred chart and describe a method for finding
 - a. 10 more than a number.
 - b. 10 less than a number.
 - c. I more than a number. More one space to the right.
 - d. I less than a number. Alove one specie to the left.
 - e. 11 more than a number.
 - f. 11 less than a number.
 - g. 9 more than a number.
 - h. 9 less than a number.
- 18. Twenty-five in base-five notation is 100_{five}. (Verify with colored chips.)
 - a. What is the base-five numeral for 24?
 - b. What is the numeral for 1 less than 1000_{five}?
 - c. How and where does this pattern occur in a base-ten numeration system? ** (application all)**
- 19. Explain the numeral *rstu* in base b where r, s, t, and u are single digits. $\frac{1}{2}(h^2) + \frac{1}{2}(h^2) + \frac{1}{2}(h^2) + \frac{1}{2}(h^2)$
- 20. Use some or all of the digits 2, 5, 8, and 0 to form whole numbers in base-ten to complete each of the following sentences, and then justify your choices:
 - a. When I drove home yesterday, traffic on the highway was moving about ____ miles per hour.
 - b. I spent about \$____ on textbooks this semester, or \$___ more than last semester.
 - c. I have more days to complete my essay.
 - d. My roommate eats ____ times as much as I do.
 - e. The score of yesterday's basketball game was _____ to
 - f. I have ____ checks left in my checkbook to last until next month.

- g. We're expecting the fundraiser to clear about
- h. The campus post office sells approximately _____stamps per week.

B. Deepening Understanding

- 21. Describe in words or with symbols the method you used to translate
 - a. the Egyptian numerals in Exercise 10 into Hindu-Arabic numerals.
 - **b.** the Babylonian numerals in Exercise 11 into Hindu-Arabic numerals.
 - c. the Roman numerals in Exercise 12 into Hindu—Arabic numerals.
 - d. the Mayan numerals in Exercise 13 into Hindu-Arabic numerals.
- 22. Complete the sequence by writing the numbers in the numeration system shown, and describe the system's characteristic(s) you used in order to do so.

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_	International Control of the Control		
a.	,,		
b.	98,,, 99, 100.	10 t	
c.	,,,,,,	∩II ⊸	87
	A A A		
d.	,,		
e.	,, CI XCVIII,	$\chi(\exists \chi$, C
f.		į	

C. Reasoning and Problem Solving

- 23. Recall from the explanation of base-ten blocks that 1 usually is represented by a cube 1 cm × 1 cm × 1 cm. Ten is represented by a tens stick with dimensions 10 cm × 1 cm × 1 cm. One hundred is a hundreds square with dimensions 10 cm × 10 cm × 1 cm. One thousand is a larger cube, 10 cm × 10 cm × 10 cm. If you were to continue using base-ten blocks for the next place-value positions, what shape and dimensions would the millions block have? Explain your conclusion.
- 24. In a base-ten numeration system, the numerals for the first 10 counting numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. In a base-five numeration system, the numerals for the first 10 counting numbers are 1_{five}, 2_{five}, 3_{five}, 4_{five}, 10_{five}, 11_{five}, 12_{five}, 13_{five}, 14_{five}, and 20_{five}. Record the base-five symbols for the first 100 counting numbers on a hundred chart. Compare the base-five numerals to the base-ten numerals in the same positions on another 10 × 10 chart. What generalizations can you form?
- 25. What patterns do you see on the five-by-five chart in Figure 2.32? Compare the patterns of the five-by-five chart to the base-ten numerals on the hundred chart.

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: Zero

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cprestem:

- 26. What patterns would you expect to see on an 8×8 chart for base-eight numerals? Make a chart to test your conjectures.
- 27. Imagine the thoughts and actions of the scribe who discovered the possibility of communicating quantities with marks on a clay envelope rather than with actual tokens. Write a short description of this hypothetical scene.
- (8) 28. Invent a new numeration system. Record the numerals for the whole numbers one through twenty. Give your list of new numerals to a classmate and ask that person to try to determine what the next 10 numerals would be. Record the comments the person makes while trying to figure out your numeration system. Critique your system based on the person's comments.
 - D. Communicating and Connecting Ideas
- 29. The Hamburger Stack. A famous fast-food chain placed signs at their restaurants, claiming: "Over 400 billion hamburgers have been sold." Think about how large a billion might be and estimate the height of this stack of hamburgers by choosing one of (a)-(f):
 - a. Higher than a Ferris wheel
 - b. Higher than the Empire State Building
 - c. Higher than a jet plane flying at an altitude of 7 miles
 - d. Higher than a 100-mile-high satellite in orbit
 - e. Higher than the distance to the moon
 - f. Higher than the distance to the sun

Determine a way to check your estimate.

30. Some numeration systems undoubtedly were developed around the idea of groups of ten because of human beings' 10 fingers. If only one hand's worth of fingers had been used, the Hindu-Arabic numeration system might have been based on groups of five. With a partner or small group, brainstorm advantages and disad-

Chapter Summary

Reflecting on Big Ideas and Essential Understandings: Questions and Answers

SECTION 2.1

What set ideas help with understanding whole numbers? (pp. 58-63) A set is a collection of objects or ideas. It may contain from zero elements (the null or empty set) to infinitely many elements. Two sets that have a one-toone correspondence are equivalent sets, and two equivalent sets that have exactly the same elements are equal sets. What are whole numbers? (pp. 64–65) A whole number is the number idea embodied in a collection of equivalent sets. Counting, or the establishment of a one-to-one correspondence between a set of number names and the objects in a set, is used to identify the whole number represented by a set.

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vantages to the base-ten and base-five numeration systems. Be prepared to defend either system as being "the best." Participate in a class debate.

- 31. Historical Pathways. For several hundred years throughout the region now known as Europe, the Roman numeration system was used to record whole numbers. Leonardo of Pisano, author of Liber abbaci, or Book of Calculation, published in 1202, is credited with the introduction of the Hindu-Arabic numeration system into European culture. What conjectures can you make about the acceptance of the Hindu-Arabic system when it was introduced in early Europe? [You might want to do some research in books by V. J. Katz (1993) or C. D. Boyer (1991) to test your conjectures.] With a small group of classmates, write a short script to show how you might react if a new numeration system were introduced in the United States. Among
 - 32. Making Connections. A numeral for the quantity zero came fairly late in the development of mathematical symbolism. Make a list of situations in which using a symbol for zero would be desirable. What difficulties would you encounter if there were no symbol for zero?
 - 33. Making Connections. The numeration system most commonly encountered in everyday experiences is based on groups of ten. However, important technological applications are based on numeration systems involving groups of two, eight, and sixteen. For example, the two digits used in a base-two numeration system can be represented by an electrical circuit being off or on. Electrical circuits in a computer can then be combined to represent any whole number quantity in base-two. As 8 and 16 are powers of two, these bases might also be useful in computers. How might the baseten number 421 be represented in (a) base-eight? (b) base-sixteen? (a) $\alpha \otimes_{\text{end}_{\mathcal{A}}}$ (b) $\{A\beta_{\text{total}}, \text{ where A represents to$

How can set ideas help in comparing and ordering whole numbers? (pp. 66–68) For whole numbers a and b and sets A and B, where n(A) = a and n(B) = b, a < bif and only if A is equivalent to a proper subset of B.

What are some important subsets of whole numbers? (pp. 68-70) The set of whole numbers contains special subsets, such as the set of natural numbers, the set of even numbers, and the set of odd numbers. An infinite set is distinguished from a finite set by being shown to be equivalent to a proper subset of itself, as the set of whole numbers is equivalent to the set of even numbers.

SECTION 2.2

How can models and set ideas be used to define addition? (pp. 72-78) Discrete models of sets and

continu used to ing pro and n(B)a + b =What a 78-80) [erty, the the assoc tion is 0. there is a How ca whole n continuo areas can bers as a How ca (pp. 82-8only if c: Thus sul missing a uend, b is Does st addition bers does associativ

SECTION 2. How can tiplication els of sets can be use as a proce Multiplica counting (gle whose finding the product of and n(B): $a \times b = c$ What are 93–96) Mi property, 1 erty, the : The idea Multiplica tributive p How can whole nu can be use two division equivalent measureme equivalent

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define ets and continuous models, including number lines, can be used to represent addition of whole numbers as a joining process. For two disjoint sets A and B, if n(A) = a and n(B) = b, then $a + b = n(A \cup B)$. In the equation, a + b = c, a and b are addends and c is the sum.

- What are some basic properties of addition? (pp. 78–80) Addition of whole numbers has the closure property, the identity property, the commutative property, and the associative property. The identity element for addition is 0. For whole numbers a and b, a < b if and only if there is a whole number k > 0 such that a + k = b.
- How can models be used to explain subtraction of whole numbers? (pp. 80–82) Discrete models of sets and continuous models including number lines, lengths, and areas can be used to represent subtraction of whole numbers as a taking away, separating, or comparing process.
- How can addition be used to define subtraction? (pp. 82–83) For whole numbers a and b, a b = c if and only if c is a unique whole number such that c + b = a. Thus subtraction may also be thought of as finding the missing addend. In the equation a b = c, a is the minuend, b is the subtrahend, and c is the difference.
- Does subtraction have the same properties as addition? (pp. 83–85) No. Subtraction of whole numbers doesn't have the closure, identity, commutative, or associative properties.

SECTION 2.3

- How can models and set ideas be used to define multiplication of whole numbers? (pp. 89-93) Discrete models of sets and continuous models, including number lines, can be used to represent multiplication of whole numbers as a process of joining equivalent sets or equal lengths. Multiplication may be represented as repeated addition, counting elements in an array, finding the area of a rectangle whose dimensions are the numbers being multiplied, or finding the number of ordered pairs created in a Cartesian product of two sets. If A and B are finite sets with n(A) = a and n(B) = b, then $a \times b = n(A \times B)$. In the equation, $a \times b = c$, a and b are factors and c is the product.
- What are some properties of multiplication? (pp. 93–96) Multiplication of whole numbers has the closure property, the identity property, the commutative property, the associative property, and the zero property. The identity element for multiplication is 1. Multiplication and addition are both involved in the distributive property of multiplication over addition.
- How can models be used to explain division of whole numbers? (pp. 96–98) Discrete models of sets can be used to represent the processes of division. The two division situations consist of finding how many equivalent subsets in a set (repeated subtraction or measurement) and finding how many elements in each equivalent subset (sharing or partitioning). Continuous

models, including lengths and areas, can also be used to show these two situations.

- How can multiplication be used to define division? (p. 99) Looking at division as the inverse of multiplication leads to the use of multiplication to define division. For whole numbers a and b, $a \div b = c$ if and only if c is a unique whole number such that $c \times b = a$. In the equation $a \div b = c$, a is the dividend, b is the divisor, and c is the quotient. Division may also be thought of as finding the missing factor or as repeated subtraction.
- Does division have the same properties as multiplication? (p. 100) Division of whole numbers doesn't have the closure, identity, commutative, associative, or zero properties. The inverse relationship between addition and subtraction parallels the inverse relationship between multiplication and division. Also, the repeated addition interpretation of whole-number multiplication parallels the repeated subtraction interpretation of whole-number division.
- What is the division algorithm for whole numbers? (pp. 100-102) For whole numbers a and b, where $b \ne 0$, a division process for $a \div b$ can be used to find unique whole numbers q (quotient) and r (remainder) such that $(q \times b) + r = a$, and $0 \le r < b$.

SECTION 2.4

- What is a numeration system? (p. 106) A numeration system is an accepted collection of properties and symbols that allows systematic writing of numerals to represent numbers.
- How can grouping help explain the Hindu-Arabic place-value numeration system? (pp. 106–113) The meaning of a place-value numeral such as 123 can only be determined by knowing the grouping system on which it is based. For any base b > 3, 123 represents 1 group of b^2 plus 2 groups of b plus 3 ones. The Hindu-Arabic numeration system is a place-value system that uses a base of ten.
- What were some other early numeration systems? (pp. 113–120) Some early numeration systems involved tokens and tallying. The early Egyptian numeration system used picture symbols based on groups of ten and tallying. The early Babylonian numeration system used two symbols based on groups of ten and place value based on groups of sixty. The Roman numeration system used a combination of tallying and placement of seven basic symbols. The Mayan system used three symbols, including one for zero, and vertical place value based mainly on groups of twenty.
- How is the Hindu–Arabic system like and unlike other systems? (pp. 120–122) The Hindu–Arabic system uses 10 symbols, including a symbol for zero, and place value based on groups of ten.

Terms, Concepts, and Generalizations

SECTION 2.1

Set (p. 58)

Element (p. 58)

One-to-one correspondence (p. 59)

Equal sets (p. 60)

Equivalent sets (p. 61)

Subset of a set (p. 61)

Venn diagram (p. 62)

Proper subset of a set (p. 62)

Universal set (p. 62)

Empty, or null, set (p. 62)

Complement (p. 63)

Finite set (p. 64)

Infinite set (p. 64)

Whole number (p. 65)

Cardinality (p. 65)

Set of whole numbers (p. 65)

Counting (p. 65)

Comparing (p. 66)

Less than (p. 67)

Greater than (p. 67)

Ordering (p. 68)

Set of natural numbers (p. 68)

Set of even numbers (p. 68)

Set of odd numbers (p. 68)

SECTION 2.2

Joining (p. 73)

Union of two sets (p. 74)

Intersection of two sets (p. 75)

Disjoint sets (p. 75)

Addition of whole numbers (p. 76)

Addend (p. 76)

Sum (p. 76)

Closure property of addition (p. 78)

Identity property of addition (p. 78)

Commutative property of addition (p. 79)

Associative property of addition (p. 79)

Less than (p. 80)

Greater than (p. 80)

Taking away (p. 80)

Separating (p. 80)

Comparing (p. 80)

Subtraction as the inverse of addition

Inverse operations (p. 82)

Subtraction of whole numbers (p. 82)

Minuend (p. 82)

Subtrahend (p. 82)

Difference (p. 82)

Missing addend (p. 83)

SECTION 2.3

Rectangular array (p. 89)

Area model (p. 90)

Repeated addition (p. 90)

Ordered pair (p. 90)

Cartesian product (p. 90)

Multiplication of whole numbers

(p. 91)

Factor (p. 91)

Product (p. 91)

Closure property of multiplication

Identity property of multiplication

(p. 94)

Commutative property of multiplication (p. 94)

Associative property of multiplication

Zero property of multiplication (p. 94)

Distributive property of multiplication over addition (p. 94)

Sharing, or partitioning, model of division (p. 96)

Repeated subtraction, or measurement, model of division

(p. 97)

Division as the inverse of multiplication (p. 99)

Division of whole numbers (p. 99)

Dividend (p. 99)

Divisor (p. 99)

Quotient (p. 99)

Missing factor (p. 99)

Division algorithm (p. 101)

Remainder (p. 101)

SECTION 2.4

Numeral (p. 106)

Numeration system (p. 106)

Hindu-Arabic numeration system

(p. 106)

Place value (p. 106)

Base (p. 107)

Proportional models (p. 110)

Nonproportional models (p. 110)

Expanded notation (p. 110)

Periods (p. 112)

Token system (p. 113)

Tally system (p. 114)

Egyptian numeration system (p. 114)

Babylonian numeration system

(p. 115)

Roman numeration system (p. 118)

Mayan numeration system (p. 118)

Zero (p. 121)

Chapter Review

Concepts and Skills

- 1. Use sets to show the following ideas:
 - **a.** The meaning of 7 $\{1, @, #, \$, \%, x, y\}$
 - **b.** The meaning of $0 \in \mathbb{R}$
 - c. 15 > 9
 - **d.** $0 < \text{all other whole numbers} \implies$
- 2. Translate Hindu-Arabic numerals 100, 66, 247, and 180 into the equivalent Egyptian, Babylonian, Roman, and Mayan numerals. (Hint: Make a table and record your answers in it.)
- 3. Describe the important mathematical characteristics of each system.
 - a. The Egyptian numeration system
 - **b.** The Babylonian numeration system
 - c. The Roman numeration system
 - **d.** The Mayan numeration system
- e. The Hindu–Arabic numeration system 4. Express the quantity 184 as the equivalent numeral in each base given.
 - a. Base five 1214
- **b.** Base two 10111000...

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c. exp: d. exp: 6. For set

> $A = \{2$ $C = \{2,$ a. AU

> b. C c. A∩

d. (Ar e. $A\cap$ f. $A\cap$

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> that can a. Takin

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f. Comp 9. For each that can b

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subsets d. Separat lengths

Reasoning ar

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5. Represent 2,045 with

a. base-ten blocks.

b. colored chips.

c. expanded notation without exponents.

d. expanded notation with exponents.

6. For sets $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

 $A = \{2, 3, 5, 6\}, B = \{3, 4\},$ and

 $C = \{2, 6, 7, 8\}, \text{ find }$

a. $A \cup B$

b. \overline{C} {0, 1, 3, 4, 5, 9, 10, 11, 12}

c. $A \cap B \cap C$

d. $(A \cap C) \cup B \ \{2, 3, 4, 6\}$

e. $\widehat{A} \cap \overline{C} \in [3, 5]$

f. $A \cap C \ (2,6)$

7. For each of the following actions, write a word problem that can be solved with the equation 85 + 62 = n:

a. Joining lengths Answers wary.

b. Joining sets Apparent vary

8. For each of the following actions, write a word problem that can be solved with the equation 85 - 62 = n:

a. Taking away one length from another

b. Taking away a subset from a set

c. Separating a length into two lengths

d. Separating a set into two subsets

e. Comparing two lengths

f. Comparing two sets Answers very (e)- (f).

9. For each of the following actions, write a word problem that can be solved with the equation $12 \times 25 = n$:

a. Joining equal lengths

b. Joining equivalent sets

c. Finding the area of a rectangle

d. Finding the number of ordered pairs

10. For each of the following actions, write a word problem that can be solved with the equation $125 \div 25 = n$:

a. Separating a set into equivalent subsets of a given size

b. Separating a length into equal lengths of a given size

c. Separating a set into a given number of equivalent

d. Separating a length into a given number of equal lengths

Reasoning and Problem Solving

11. Flower Arrangements. The final assignment for 15 horticulture students is to design a flower arrangement by using, at most, four different types of flowers. Can the professor reasonably ask that no student use the same combination of flowers as another student uses? Why or why not?

- 12. Use expanded notation to describe the value of $1,045_b$.
- 13. Use the definition of subtraction to justify that 180 59 = 121.
- 14. Use addition to justify that 123 > 85.

15. Consider the set $S = \{0, 10, 20, 30, ...\}$. Support your answer to each of the following questions:

a. Is S closed under addition?

b. Does S have an additive identity?

c. Is S closed under multiplication?

d. Does S have a multiplicative identity?

e. How do the other properties of addition and multiplication compare between set *S* and the set of whole numbers?

16. Use the definition of division to justify that $120 \div 40 = 3$. 120 and $0 \approx 3$ because $3 \times 40 \approx 120$

17. Use the definition of the division algorithm to justify that $125 \div 40 = 3$ with a remainder of 5.

18. Which Answer Is Right? Marcy and two friends had each written a division word problem that could be solved with the equation 18 ÷ 5 = ____. One friend argued that the answer to her problem was 3 and the other argued that his answer was 4. Marcy believed that the answer to her problem was 3 with a remainder of 3. Write a paragraph explaining how all three could be right.

Answers vary

Alternative Assessment

19. Draw a diagram or concept map explaining the connections between the ideas of one-to-one correspondence, equivalent sets, equal sets, and whole numbers.

20. The ancient Incas used a *quipu*, an arrangement of colored and knotted cords tied to a base rope, as a way to record numbers. Design a numeration system based on this idea of knotted cords. Write a description of your system and compare its characteristics to those of the Hindu–Arabic system. Answers vary.

21. Choose three classmates and form a group. Each person is to prepare one of the following comparisons of two whole-number operations to present to the group: addition and subtraction, addition and multiplication, multiplication and division, or subtraction and division. Write a summary of the main ideas in your group's presentations.

22. Many students have difficulty memorizing basic arithmetic facts to the point of having immediate recall. Use the properties of addition and multiplication and outline a set of strategies that would be helpful for students as they learn basic arithmetic facts. Answers very.