37. Use the four elements a, b, c, and d and generate an operation, #, that has all four addition properties. Describe the operation in a table like that in Exercise 36.

38. Return to the picture you drew in Mini-Investigation 2.5 about how addition and subtraction are different and how addition and subtraction are alike. What changes would you make to this diagram now? Explain why. Answers vary.

39. Return to the definition of *less than* for whole numbers.
   a. Why is $k > 0$ an important part of the definition? What would happen if it were omitted?
   b. How would a definition of *greater than* based on whole-number operations and the properties of these operations differ from the definition of *less than?*

D. Communicating and Connecting Ideas

40. Because subtraction is defined by using addition, you could argue that subtraction isn't an operation in its own right—that all joining, separating, and comparing situations could be described with an addition equation. With two or three other classmates, present an argument orally or in writing to support subtraction as an operation having its own definition and symbolism.

41. **Historical Pathways.** When recording on papyrus, the Egyptians used a set of mathematical symbols that included separate symbols for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and so on. For example, the symbol for 432 was written with the symbols for 400, 30, and 2 lined up next to each other. Addition with these symbols probably required the use of extensive addition tables showing the sums of each pair of symbols. How might the scribes have dealt with subtraction?

42. **Making Connections.** For many pairs of whole numbers, a and b, no whole number represents $a - b$. That is, whole numbers are not closed under the operation of subtraction. Based on your earlier experiences in mathematics, describe the kinds of numbers that, when included with whole numbers, bring closure to subtraction. What physical situations could these numbers model? How would these numbers be related to the set of whole numbers? How could these numbers be helpful when you use subtraction to compare two whole numbers?

Section 2.3

Multiplication and Division of Whole Numbers

Using Models and Sets to Define Multiplication • Properties of Multiplication • Modeling Division • Using Multiplication to Define Division • Comparing Division to Multiplication • The Division Algorithm for Whole Numbers

In this section, we use models to help explain multiplication and division of whole numbers and the actions involved in these operations. We also examine the mathematical definitions and properties of the operations of multiplication and division and investigate the relationship between these operations.

Essential Understandings for Section 2.3

- Some real-world problems involving joining equal groups, separating equal groups, comparison, or combinations can be solved using multiplication; others can be solved using division.
- Multiplying by x is the inverse of dividing by x (when $x \neq 0$).
- Properties of whole numbers apply to certain operations but not others.
- Two numbers can be multiplied in any order.
- The product of any nonzero number and 1 is the number.
- Three or more numbers can be grouped and multiplied in any order.

Mini-Investigation 2.7 asks you to compare the operations of multiplication and division.

Draw a picture to illustrate your comparison of multiplication and division.

MINI-INVESTIGATION 2.7 **Making a Connection**

How are multiplication and division different, and how are they alike?

Using Models and Sets to Define Multiplication

Models for multiplication are similar to models for addition. Addition can be modeled by joining two sets, and multiplication can be modeled by joining a certain number of equivalent sets. When these sets are arranged in equal rows and columns, as shown in Figure 2.20, this arrangement of the three equivalent sets is called a **rectangular array.** In any multiplication model involving discrete objects, one number represents the number of sets (rows) and one number represents the number of objects in each set (row). The idea behind these models and the language of sets is used in the following description of multiplication:

> $a \times b$ is the number of elements in the union of a equivalent sets, each containing b elements.

Addition also can be modeled by joining two directed segments along the number line, and multiplication can be modeled by joining a certain number of directed segments of equal length, as illustrated in Figure 2.21. In any multiplication model involving lengths, one number represents the number of segments being joined and one number represents the length of one segment.

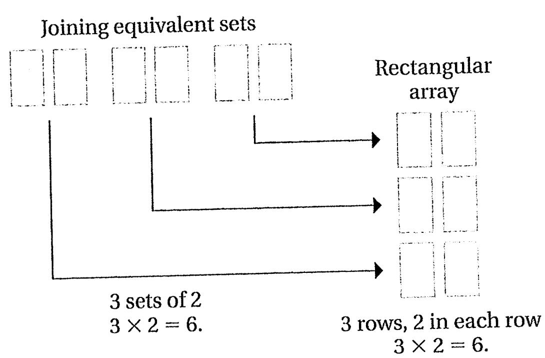


FIGURE 2.20
Multiplication as joining equivalent sets and as a rectangular array.

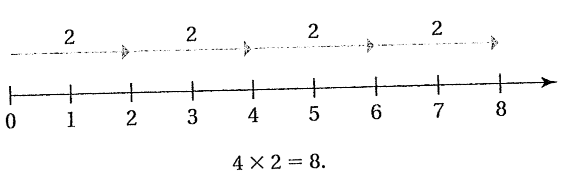
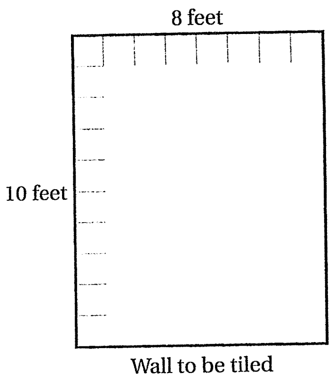


FIGURE 2.21
Multiplication as joining segments of equal length on a number line.



$10 \times 8 = 80.$

FIGURE 2.22
Multiplication as the area of a rectangle.

Area Model. In an **area model** of whole-number multiplication, the two numbers being multiplied represent the dimensions of a rectangle. The area of the rectangle is the result of the multiplication. For example, if you are decorating 1-foot-square tiles to cover a rectangular wall that is 10 feet high and 8 feet wide, you naturally ask, "How many tiles do I need?" When you answer this question, you are demonstrating a fundamental meaning of multiplication, as illustrated in Figure 2.22.

The area model is similar to the array model because the area of a rectangle with whole-number length and width can be thought of as determining the total number of unit squares when the number in each row and the number of rows are known. However, the area and array models are different in that the rectangular dimensions aren't always whole numbers and in that the array model deals with discrete objects, whereas the area model deals with a continuous region.

Using Repeated Addition to Define Multiplication. The idea that 5×3, for example, can be found by putting together 5 sets with 3 objects in each set suggests that 5×3 can be interpreted as $3 + 3 + 3 + 3 + 3$. Thus we sometimes say that multiplication is *repeated addition.*

Definition of Multiplication as Repeated Addition

In the **multiplication of whole numbers,** if there are m sets with n objects in each set, then the total number of objects ($n + n + n + \cdots + n$, where n is used as an addend m times) can be represented by $m \times n$, where m and n are **factors** and $m \times n$ is the **product.**

Using the Language of Sets to Define Multiplication. As described earlier, we can use set language to complete a description of $a \times b$ as the number of elements in the union of a disjoint equivalent sets, each containing b elements. Although this description provides a possible definition of multiplication of whole numbers, let's consider another set idea that can be used to define multiplication, called the *Cartesian product* of two sets. An example of the idea of an **ordered pair** used in the definition is (cat, dog), which is a different ordered pair than (dog, cat).

Definition of Cartesian Product

The **Cartesian product** of two sets A and B, $A \times B$ (read "A cross B") is the set of all ordered pairs (x, y) such that x is an element of A and y is an element of B.

For example, if $A = \{1, 2, 3\}$ and $B = \{p, q\}$, then $A \times B = \{(1, p), (1, q), (2, p), (2, q), (3, p), (3, q)\}$. Note that sets A and B can be equal. Example 2.11 illustrates the idea of a Cartesian product of a set with itself.

Example 2.11 **Illustrating a Cartesian Product**

In a particular game of chance, a player's turn consists of rolling a die twice. What are the possible results a player could get on a turn? How many results are there?

SOLUTION

A player could roll a 1 first, then a 1, 2, 3, 4, 5, or 6 on the second roll. Similarly, if 2 came up on the first roll, the second roll could be 1, 2, 3, 4, 5, or 6. The entire set of possibilities may be represented in the diagram on p. 91. Each die can be modeled by a set of six numbers, $S = \{1, 2, 3, 4, 5, 6\}$. The 36 resulting pairs of numbers represent the Cartesian product, $S \times S$.

Number on first roll \ Number on second roll	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

YOUR TURN

Practice: In another game, a player's turn consists of two spins of a spinner, with options 0, 1, 2, or 3 on the spinner. What are the possible results a player could get on a turn? How many results are there?

Reflect: What generalization can you discover from the two problems in this example? ■

We now use the idea of Cartesian product to give a commonly used definition of multiplication of whole numbers.

Definition of Multiplication of Whole Numbers

In the **multiplication of whole numbers,** if A and B are finite sets with $a = n(A)$ and $b = n(B)$, then $a \times b = n(A \times B)$. In the equation $a \times b = n(A \times B)$, a and b are called **factors** and $n(A \times B)$ is called the **product.**

The essential idea of the definition is that $a \times b$ is the number of elements in the Cartesian product of two sets, one set containing a elements and the other set containing b elements. Multiplication as the representation of a Cartesian product can occur in a variety of physical situations. As illustrated by Example 2.12, they include counting combinations and rectangular arrays. Useful problem-solving strategies include "act it out" or "draw a diagram."

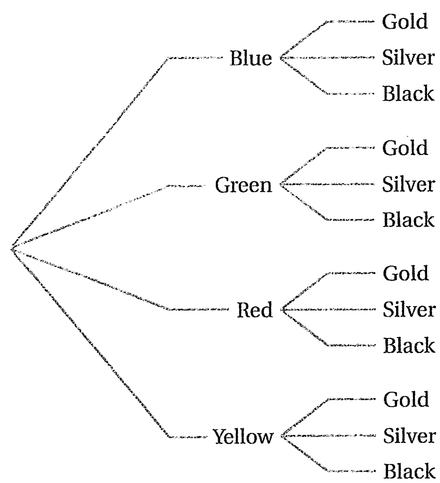
Example 2.12 Problem Solving: Color Combinations for Invitations

Suppose that you are using construction paper to make invitations for a club function. The construction paper comes in blue, green, red, and yellow, and you have gold, silver, or black ink. How many different color combinations of paper and ink do you have to choose from?

Working Toward a Solution

Understand the problem	*What does the situation involve?*	Selecting colors of paper and colors of ink for invitations.
	What has to be determined?	The number of different combinations.
	What are the key data and conditions?	Four colors of paper; three colors of ink.

	What are some assumptions?	Each invitation combines one color of paper and one color of ink.
Develop a plan	*What strategies might be useful?*	Act it out or draw a diagram.
	Are there any subproblems?	Find the number of different invitations that can be made with blue paper.
	Should the answer be estimated or calculated?	Calculate the exact number.
	What method of calculation should be used?	Use the diagram to count.
Implement the plan	*How should the strategies be used?*	Use a choice, or tree, diagram to match each possible choice of paper with each choice of ink.



	What is the answer?	Four kinds of paper, each with three colors of ink, make 12 combinations.
Look back	*Is the interpretation correct?*	*Check:* The diagram fits the problem.
	Is the calculation correct?	*Check:* The count is correct.
	Is the answer reasonable?	Yes. Three choices of ink for each color of paper would triple the kinds of invitations made with one color of ink.
	Is there another way to solve the problem?	As shown, you can use an array of ordered pairs to list the kinds of invitations. Or you can use a multiplication equation: $4 \times 3 = 12$ combinations. In the array of ordered pairs, each color of paper is matched with each color of ink in an ordered pair that describes an invitation.

	Gold	**Silver**	**Black**
Blue	(B, G)	(B, S)	(B, Bk)
Green	(GR, G)	(GR, S)	(GR, Bk)
Red	(R, G)	(R, S)	(R, Bk)
Yellow	(Y, G)	(Y, S)	(Y, Bk)

YOUR TURN

Practice: From a committee, one woman's name will be drawn and one man's name will be drawn to see who serves on a community advisory board. If 9 women and 11 men are on the committee, how many different possible combinations of one woman and one man are there?

Reflect: What solution to the practice problem is suggested by the Cartesian product definition of multiplication? ■

Connection to the PreK-8 Classroom

Although multiplication is usually first introduced in the early grades as joining equivalent sets or as repeated addition, young students need to experience the other actions connected to multiplication. These experiences include looking for all possible pairs, finding the number in an array, and finding the number of tiles in a rectangle. They provide for a broader understanding of the concept of multiplication and its use in counting outcomes for sample spaces in probability and in determining areas and volumes of surfaces and solids.

Properties of Multiplication

As the pirate in the cartoon in Figure 2.23 seems to know, multiplication, like addition, is commutative. The properties that are important in addition—the closure property, the identity property, the commutative property, and the associative property—also hold for multiplication.

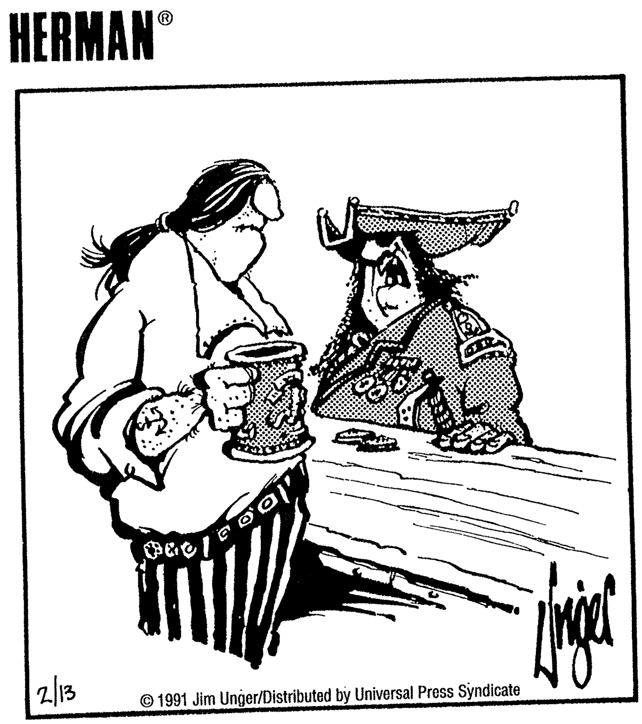


FIGURE 2.23
"Can you change two pieces of eight for eight pieces of two?"

We summarize the properties of multiplication as follows:

Properties of Multiplication of Whole Numbers

Closure property	For whole numbers a and b, $a \times b$ is a unique whole number.
Identity property	There exists a unique whole number, 1, such that $1 \times a = a \times 1 = a$ for every whole number a. Thus 1 is the *multiplicative identity element.*
Commutative property	For whole numbers a and b, $a \times b = b \times a$.
Associative property	For whole numbers a, b, and c, $(a \times b) \times c = a \times (b \times c)$.
Zero property	For each whole number a, $a \times 0 = 0 \times a = 0$.
Distributive property of multiplication over addition	For whole numbers a, b, and c, $a \times (b + c) = (a \times b) + (a \times c)$.

Figure 2.24 shows one way to verify the commutative property of multiplication.

The properties of multiplication different from the properties of addition include a different identity element for the identity property and the zero property. The distributive property of multiplication over addition connects the two operations.

Identity Property of Multiplication. According to the **identity property of multiplication,** there is a specific whole number, *one*, that when multiplied by a whole number, n, leaves n as the product:

One is the unique whole number such that for each whole number a, $a \times 1 = 1 \times a = a$.

Thus, 1 is called the *multiplicative identity element.*

Zero Property of Multiplication. The **zero property of multiplication** states that, whenever 0 is used as a factor, the product is always 0:

For each whole number a, $a \times 0 = 0 \times a = 0$.

This property is important because 0 can often hide the value of another number in multiplication. For example, in the equation $a \times b = 0$, a could be any number if $b = 0$. The zero property of multiplication also leads to an important conclusion about division developed on p. 99.

Distributive Property of Multiplication over Addition. The **distributive property of multiplication over addition** connects the operations of multiplication and addition. In general,

For whole numbers a, b, and c, $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$.

This property may be used to generate unknown multiplication facts from known ones and may be represented by looking at an array model from two different viewpoints, as shown in Figure 2.25.

Connection to the PreK–8 Classroom

Analyzing a Textbook Page: The Commutative Property

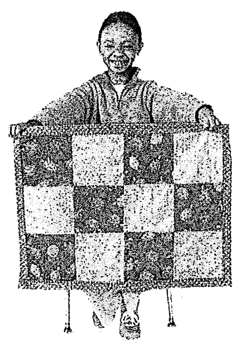
The commutative property of multiplication is a useful property for learning related basic facts. The following page in an elementary textbook shows a real-world model for the commutative property of multiplication.

Activity

Does it matter in what order you multiply?

Lynn multiplies two different ways to find how many squares are in her quilt.

First she holds her quilt like this.



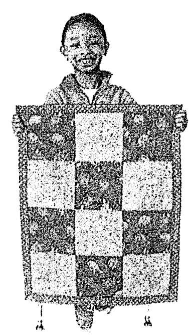
X X X X
X X X X
X X X X

3 rows, 4 in each row

$4 + 4 + 4 = 12$

$3 \times 4 = 12$

Then she holds the quilt like this.



X X X
X X X
X X X
X X X

4 rows, 3 in each row

$3 + 3 + 3 + 3 = 12$

$4 \times 3 = 12$

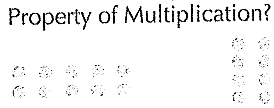
There are 12 squares in Lynn's quilt.

The **Commutative (order) Property of Multiplication** says you can multiply in any order and the product is the same.

So, $3 \times 4 = 4 \times 3$.

a. Make two arrays with counters to show that 3×2 has the same product as 2×3. Write a multiplication sentence for each array you made.

b. How do the arrays below show the Commutative Property of Multiplication?

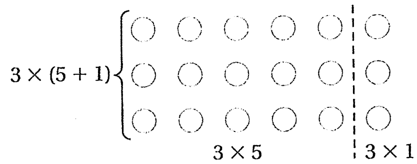


Think It Through
I can **use objects** to show that I can multiply in any order.

FIGURE 2.24

Source: *Scott Foresman–Addison Wesley Mathematics, Grade 3,* p. 263. © 2004 Pearson Education. Reprinted with permission.

What other real-world models of the commutative property of multiplication can you think of?



$$3 \times (5 + 1) = (3 \times 5) + (3 \times 1).$$

FIGURE 2.25
Array model of the distributive property of multiplication over addition.

Connection to the PreK-8 Classroom

The identity property, commutative property, zero property of multiplication, and the distributive property of multiplication over addition may be used to help students learn multiplication facts. Of the 100 products with factors of 0–9, 19 have a factor of 0, and 17 more have a factor of 1. Seeing the patterns caused by the zero and identity properties automatically allows students to know 36 of the basic multiplication facts. Of the other 64 products, 8 are doubles ($a \times a$), leaving 56 products. The commutative property can be used to help students remember half of these products once the other half has been learned. For example, if a student knows 7×8, the student then knows that 8×7 is the same. The remaining 28 products can be connected to known products by using the associative or distributive properties. For example, 6×7 can be thought of as $(6 \times 6) + 6$, an application of the distributive property of multiplication over addition, and 5×4 can be thought of as $(5 \times 2) \times 2$, an application of the associative property of multiplication.

Modeling Division

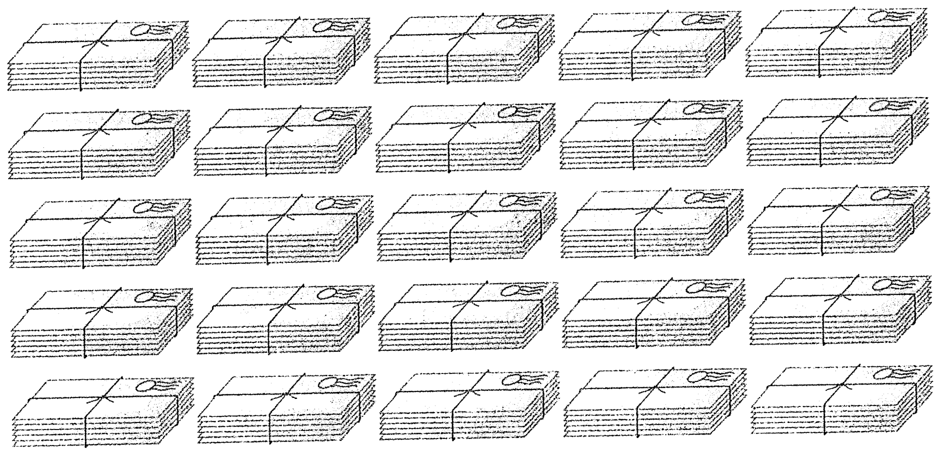
Division of whole numbers may be modeled by separating a set of objects into equivalent subsets and asking appropriate questions. Division may also be represented with continuous models such as lengths and areas, but we focus primarily on the set models.

Finding-How-Many-in-Each-Subset Model. One model for division involves separating a set into a known number of equivalent subsets and asking the question, "How many are in each subset?" The following real-world situation illustrates this model. Suppose that the envelopes containing the 300 invitations to a club function need to be bundled for distribution and that 25 large rubber bands are available to wrap around the bundles. When you bundle the set of 300 envelopes into 25 bundles and ask the question, "How many invitations are in each bundle?" you are demonstrating a fundamental meaning of division, as illustrated in Figure 2.26.

If you think of starting with the 300 envelopes and 25 rubber bands, you could share envelopes with the rubber bands by placing one envelope in each rubber band in turn until all the envelopes are distributed. Then you could count the number held by each rubber band. This approach suggests a **sharing, or partitioning, model of division.**

Finding-How-Many-Subsets Model. A second model for division involves separating a set into equivalent subsets of known size and asking the question, "How many subsets?" Figure 2.27 illustrates this model. To understand Figure 2.27, suppose that 300 invitations to a club function have been placed in envelopes

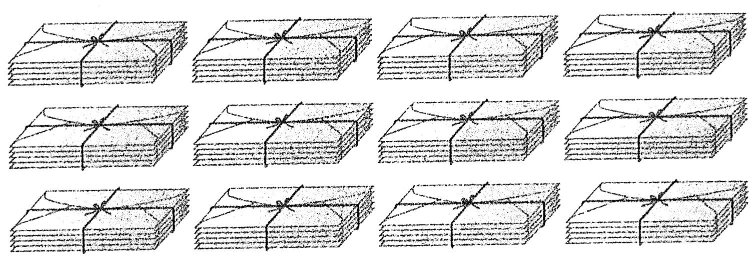
A total of 300 envelopes, with 25 bundles. How many in each bundle?



300 ÷ 25 = 12.

FIGURE 2.26
Partitioning model: Division as separating and finding how many in each subset.

A total of 300 envelopes, with 25 in each bundle. How many bundles?



300 ÷ 25 = 12.

FIGURE 2.27
Measurement model: Division as separating and finding how many subsets.

that must be bundled for distribution. A stack of 25 envelopes seems to be convenient for bundling (and easy to count when combining bundles). When you bundle the set of 300 envelopes with 25 in each bundle and ask the question, "How many bundles are there?" you are demonstrating a fundamental meaning of division.

If you think about removing bundles of 25 to put in a mailbag, you could ask the question, "How many bundles (subsets) of 25 can be taken away from 300?" This question suggests a **repeated subtraction, or measurement, model of division.** You could think about taking away bundles (units) of 25 envelopes at a time and write $300 - 25 - 25 - 25 - \cdots = 0$. By counting the number of 25s subtracted to end up with 0 envelopes, you find the number of bundles that can be made and at the same time show that $300 \div 25 = 12$.

Mini-Investigation 2.8 is intended to help you extend your understanding of these two models for division of whole numbers.

Talk about which of the division models represents a process of one for Jim, one for Jan, one for Sue, and so on, that you might use to share some items with some friends.

MINI-INVESTIGATION 2.8 Solving a Problem

What two problems can you state that can be solved by using the division 24 ÷ 6 and that illustrate the two models of division described in Figures 2.26 and 2.27?

Example 2.13 provides a brief look at how continuous models rather than discrete set models might be used.

Example 2.13 Problem Solving: Art Supplies

Solve the problems in parts (a) and (b) and indicate which problem is a continuous counterpart of the *finding-how-many-subsets* model and which is a continuous counterpart of the *finding-how-many-in-each-subset* model:

a. Mrs. Chance has a roll of butcher paper 50 feet long. She wants to cut it into 2-foot lengths for her students to use in art class. Will she have enough for her class of 22 students?

b. Mrs. Chance has a roll of butcher paper 50 feet long. Only 10 students want to work with butcher paper today. If she wants to distribute the entire length of paper equally to the 10 students, how much paper should she give each of them?

SOLUTION

a. Mrs. Chance wants to find the number of 2-foot pieces of paper in a 50-foot roll, so she divides: 50 ÷ 2 = 25, or enough paper for 25 students. Because she knows the total length and the length of one piece and is trying to find the number of pieces, this problem is similar to *finding how many subsets.*

b. Mrs. Chance has 50 feet of paper and wants to find the length of a piece that she can cut for each of 10 students, so she divides: 50 ÷ 10 = 5, or 5 feet of paper for each student. Because she is trying to find the length of each piece, this problem is similar to finding how many in each subset.

YOUR TURN

Practice: Solve problems (a) and (b), and tell which model of division applies to each solution:

a. Rikki has 36 feet of ribbon to use in wrapping presents. If each present takes 4 feet, how many presents can she wrap with the ribbon?

b. Rikki has 36 feet of ribbon to use in wrapping presents. If she has four presents to wrap, what is the maximum amount of ribbon she could use for each present?

Reflect: In general, what question does division answer when you know the total length of a strip and are given the following information?

a. the length of each piece

b. the number of equal pieces to be cut ■

Using Multiplication to Define Division

Division as the Inverse of Multiplication. The following real-world situation helps explain the special relationship between multiplication and division. How many tables that seat 8 people will you need to provide banquet seating for 72 people? You might naturally first think about the number sentence $? \times 8 = 72$. However, to find the missing number of tables, you might actually complete the division $72 \div 8 = ?$ Note that the answer to the division in the second number sentence is the missing factor in the first number sentence. This situation illustrates the idea that multiplication, represented by putting several equivalent sets together, and division, represented by separating a set into several equivalent subsets, are inverse operations. That is, in doing multiplication, you multiply two factors to find a product, whereas in doing division, you divide the product by one factor to get the other factor:

Factor		Factor		Product
9	$\times$	8	$=$	72

Product		Factor		Factor
72	$\div$	8	$=$	9

So the answer to a division sentence is one of the factors in the related multiplication sentence. This outcome suggests the following definition:

Definition of Division

In the **division of whole numbers** a and b, $b \neq 0$, $a \div b = c$ if and only if c is a unique whole number such that $c \times b = a$. In the equation, $a \div b = c$, a is the **dividend,** b is the **divisor,** and c is the **quotient.** The operation $a \div b$ may also be written as $\frac{a}{b}$ or as $b\overline{)a}$.

Division as Finding the Missing Factor. The inverse relationship between multiplication and division is the basis for a slightly different view of division: finding the missing factor. That is, in multiplication you know two factors and find the product. In division, you know the product and one factor and find the other factor:

Product		Factor		Factor
36	$\div$	3	$=$	?

When asked to find the quotient $36 \div 3 = ?$, think of 36 as the product and 3 as one of the factors. Then ask, "What factor multiplied by 3 gives the product 36?"

Why Division by 0 Is Undefined. The view of division as finding the missing factor helps you deal with the division when 0 is involved. Equations such as $8 \div 0 = ?$ have no whole number that when multiplied by 0 gives 8 or any number other than 0. For the equation $0 \div 0 = ?$ every number multiplied by 0 gives 0. Because equations such as $8 \div 0 = ?$ have no solution and the equation $0 \div 0 = ?$ has infinitely many solutions, division by 0 is said to be undefined. In other words, the definition for division cannot be applied to a divisor of 0. However, dividing 0 by another number, say, $0 \div 8$, gives the quotient 0.

Comparing Division to Multiplication

Because the operation of division is defined in terms of multiplication, a natural question is whether the properties for multiplication of whole numbers hold for division. We examine this question in Example 2.14 by looking for counterexamples for each of the properties of multiplication as they relate to division of whole numbers.

Example 2.14 Comparing Multiplication and Division Properties

Does whole-number division have the closure property?

SOLUTION

As with multiplication, division of whole numbers would have the closure property provided that when we divide two whole numbers, the quotient is a unique whole number. As $11 \div 3$ doesn't have a whole-number answer, it serves as a counterexample proving that division is *not* closed for the set of whole numbers.

YOUR TURN

Practice: Provide counterexamples showing that the commutative, associative, and distributive properties for multiplication of whole numbers don't hold for division.

Reflect: Why is 1 *not* the identity element for whole-number division? ■

The counterexamples presented in Example 2.14 demonstrate that division doesn't have the same properties as multiplication.

Now that we have explored the operations of addition, subtraction, multiplication, and division, we need to take a brief look at how they relate to one another. As Figure 2.28 shows, the inverse relationship between addition and subtraction parallels the inverse relationship between multiplication and division. Also, the repeated addition relationship between addition and multiplication parallels the repeated subtraction relationship between subtraction and division.

The Division Algorithm for Whole Numbers

Although the mathematical definition for division of whole numbers restricts its use to numbers that "come out even," as in $24 \div 4 = 6$, whole-number division is often used to model physical situations that involve making groups of equal size whereby part of a group is "left over." For example, if you are setting up a conference room for 25 people and want to have 4 people at each table, you might find $25 \div 4$ and note that you could have six full tables with 1 person left over. In such situations when we want to describe $a \div b$ $(b \neq 0)$ and no whole number c exists such that

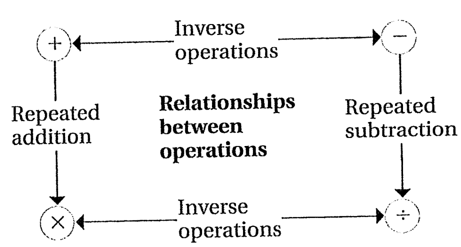


FIGURE 2.28
Relationships between operations.

$c \times b = a$, we use the **division algorithm.** Essentially, it extends the definition of division to include the possibility of a **remainder.**

The Division Algorithm

For any two whole numbers a and b, $b \neq 0$, a division process for $a \div b$ can be used to find unique whole numbers q (quotient) and r (remainder) such that $a = bq + r$ and $0 \leq r < b$. For $a = 25$, $b = 4$, $q = 6$, and $r = 1$, $25 = (4 \times 6) + 1$.

In vertical form, we write

```
    6 remainder 1
 4)25
  -24 (4 × 6)
    1
```

We can also use a calculator with integer division capability to illustrate the division algorithm. For example, using integer division to express $25 \div 4$ may result in the following screen:

```
25 INT÷ 4
 Q=6
 R=1
```

In the symbolism of the division algorithm, $a = 25$ and $b = 4$. The calculator produces the quotient, $Q = 6$, and the remainder, $R = 1$. From the calculator display, we can conclude that $25 = (4 \times 6) + 1$. Based on this example, you should be able to predict what will be displayed on the following screen when you press ENTER and to express the data in a form described in the division algorithm:

```
145 INT÷ 7
```

Mini-Investigation 2.9 extends the meaning of the division algorithm.

Talk about how the idea of the division algorithm would change if these words were omitted from the definition.

MINI-INVESTIGATION 2.9 Using Mathematical Reasoning

Why is the phrase "and $0 \leq r < b$" included in the definition of the division algorithm for whole numbers?

Connection to the PreK-8 Classroom

For children, learning about and using operations involves more than just memorizing basic facts and properties. Using an operation also requires practical, common-sense reasoning. For example, if 27 students are going on a field trip and each car will hold 4 students, how many cars will be needed for the field trip? This situation is one of forming equivalent subsets, so the problem can be solved by using ideas from the division algorithm: $27 \div 4 = 6$ full cars and 3 students left over because $(4 \times 6) + 3 = 27$ and $0 < 3 < 4$. Students need to be able to interpret the results of the division algorithm in terms of the problem being solved. In this case, the answer to the problem is that seven cars will be needed for everyone to be able to go on the field trip.

Problems and Exercises for Section 2.3

A. Reinforcing Concepts and Practicing Skills

1. Use sets to verify the following products:
   a. $8 \times 4 = 32$ b. $5 \times 10 = 50$
   c. $6 \times 0 = 0$ Show the union of 6 empty sets.
2. Use lengths on a number line to verify the following products:
   a. $8 \times 4 = 32$ b. $5 \times 10 = 50$
   c. $6 \times 0 = 0$
3. Use a rectangular array to verify each of the following products:
   a. $8 \times 4 = 32$ b. $5 \times 10 = 50$
   c. $6 \times 0 = 0$ No array, or rectangle, of 0 length
4. Create a word problem involving joining equivalent sets to go with the equation $4 \times 30 = 120$.
5. Create a word problem involving joining segments of equal length to go with the equation $4 \times 30 = 120$.
6. Write a multiplication equation and find the area of each of the following rectangles:
   a. a rectangle 12 units high and 6 units wide
   b. a rectangle 42 units high and 1 unit wide
   c. a rectangle $\frac{b}{2}$ units high and $4b$ units wide

Write a multiplication equation for and find the answer to each of the following questions in Exercises 7–9.

7. If a furniture manufacturer has four styles of chairs and six different upholstery fabric choices for each chair, how many different chairs can be made?
8. A gift basket store sells three kinds of chocolate, five kinds of cheese, and three kinds of gourmet coffee. If the store has a special on a basket that contains one selection each of chocolate, cheese, and coffee, how many different basket specials can be made?
9. If a department store wants to advertise 64 different gift wrappings (based on a choice of one type of paper and one color of bow), how many different types of wrapping paper and colors of bows should it plan to have on hand?
10. For the equation, $25 \times 4 = 100$, create an accompanying word problem involving
    a. a Cartesian product. See Exercises 7-9 for examples.
    b. an array.
    c. the area of a rectangle.
11. Which properties verify the following conjectures? Model each situation with an appropriate equation or mathematical description.
    a. To remember 8×4, I just need to remember 4×8.
    b. It's easy to multiply with 0. It doesn't even matter what the other factor is. Zero property of multiplication
    c. I can never remember what 8×6 is. But I *can* remember that 8×5 is 40. So I mentally add on one more 8 to get 48.
    d. I almost made a mistake on my worksheet because in one place I put that 9×3 is 27 and in another place I put that it is 29. But I know that it has the same answer every time, so I checked them both to see which one was right.
    e. I like to multiply by 10 or 100. The digits just move to different place-value positions.
12. Use the distributive property of multiplication over addition to find each product:
    a. $(20 + 5) \times 3$ 75 b. $4 \times (5 + 6)$ 44
    c. $(x + 10) \times (3x + 2)$ $3x^2 + 32x + 20$

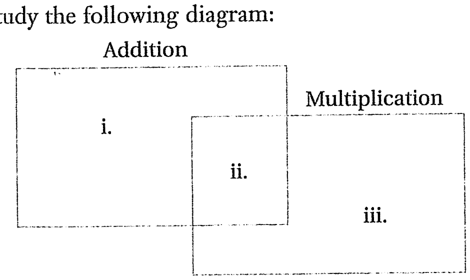
For the equation $24 \div 6 = 4$, create an accompanying word problem for each situation in Exercises 13–17.

13. Separating a set into same size equivalent sets
14. Separating a segment into same-length segments
15. Separating a set into a specific number of equivalent sets
16. Separating a segment into a specific number of segments of equal length
17. Finding the missing factor

18. Use a calculator and repeated subtraction to find each of the following quotients:
   a. $143 \div 11$ b. $108 \div 9$
   c. $1{,}404 \div 52$ Subtract 52 twenty-seven times
19. Use the definition of division to rewrite the following division equations as multiplication equations. If possible, find a whole-number value for each variable to make the statements true.
   a. $18 \div 6 = n$ b. $25 \div x = 5$
   c. $y \div 42 = 126$ d. $0 \div b = c$
20. Use the definition of division to determine the division equations that are related to each of the following multiplication equations:
   a. $15 \times 3 = n$ b. $9 \times y = 9$
   c. $r \times s = t$ d. $8 \times 0 = 0$
21. Use the division algorithm for whole numbers to find each quotient and remainder, and then justify your answers:
   a. $26 \div 3$ 8 R 2; 8(3) + 2 = 26
   b. $292 \div 21$ 13 R 19; 21(13) + 19 = 292
   c. $4 \div 7$ 0 R 4; 0(7) + 4 = 4

B. Deepening Understanding

22. Study the following diagram:



   i. Characteristics of addition that are different from multiplication
   ii. Characteristics that are common to both operations
   iii. Characteristics of multiplication that are different from addition

   a. Complete the diagram to illustrate how the characteristics of multiplication and addition of whole numbers compare.
   b. Prepare a similar diagram for subtraction and division.
23. Predict the number of ordered pairs that will be generated from the Cartesian products in (a)–(c). Then find the Cartesian products and compare the number of ordered pairs in each one to your predictions. [*Hint:* Make a three-column (predicted $n(C)$, C, and actual $n(C)$), three-row (a, b, c) table and record your answers.]
   a. $\{1, 2, 3, 4\} \times \{\,\} = C$
   b. $\{r, s, t\} \times \{a, b\} = C$
   c. $\{a\} \times \{5, 6, 7, 8, 9\} = C$
   d. Write the multiplication equation related to each of the Cartesian products.
24. Using the union of two sets to define addition requires that the sets be disjoint. Is that condition necessary for the two sets in the Cartesian product definition for multiplication? Why or why not? Give an example to support your response.
25. What role does multiplication play in our base-ten numeration system?
26. The product 12×6 can be found by thinking, $(10 + 2) \times 6 = (10 \times 6) + (2 \times 6) = 60 + 12 = 72$.
   a. What properties of multiplication are being used?
   b. Use these properties to verify the product 12×64.
27. Use the distributive property of multiplication over addition to rewrite each of the following sums as the product of two factors, where one of the factors is a sum:
   a. $4rs + 16r$ $4r(s + 4)$
   b. $27a^2 + 81a + 9$ $9(3a^2 + 9a + 1)$
   c. $24a + 14b + 20c$ $2(12a + 7b + 10c)$
28. The basic facts table for multiplication of whole numbers is as follows:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

   a. Describe how the closure property of multiplication is reflected in the table.
   b. Describe how the multiplicative identity property is reflected in the table.
   c. Describe how the commutative property of multiplication is reflected in the table.
   d. Describe how the zero property of multiplication is reflected in the table.
   e. Describe how the distributive property of multiplication over addition can be illustrated by the table.
   f. How are these properties important in learning basic multiplication facts? Answers vary.
29. How can the multiplication table be used to solve division equations?
30. Division can be used to model both separating a quantity into groups of a given size and separating a quantity into a given number of groups of equal size.
   a. How are these situations different? (For example, what different questions are asked? What different actions are represented?)
   b. How are these situations alike?

31. Consider the following statement: $(q \times 12) + r = 64$.
   a. Identify all possible pairs of whole numbers for q and r that make the sentence true.
   b. Which pairs of values for q and r satisfy the division algorithm for whole numbers? $q = 5$ and $r = 4$
   c. What could happen if the division algorithm didn't include the restriction that r is greater than or equal to 0 and less than the divisor?

32. The distributive property of division over addition has the form

For whole numbers a, b, and c, with $c \neq 0$, $(a + b) \div c = (a \div c) + (b \div c)$, provided that whole number quotients exist for each division expression.

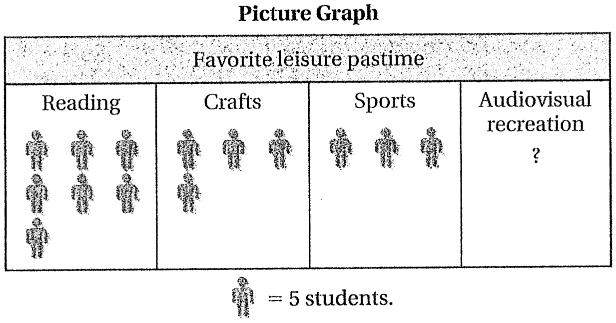
   a. Identify four sets of three whole numbers that illustrate this distributive property of division over addition.
   b. What patterns do you see in these sets of whole numbers?
   c. Why does the distributive property of division over addition work only in particular formats?

C. Reasoning and Problem Solving

33. Describe as many patterns as you can find in the multiplication facts table in Exercise 28. Use the commutative, associative, and distributive properties to explain why these patterns occur. Answers vary.

34. Consider the data shown in the following picture graph and frequency chart. Answers vary.

Picture Graph



Frequency Chart

Pastime	Reading	Crafts	Sports	Audiovisual recreation
Frequency	?	?	?	33

   a. Write two multiplication word problems and two division word problems reflecting the relationship between the data in the graph and the chart.
   b. Write an equation that represents the solution to each of your problems. Could either solution be represented by more than one equation?

35. The Candy Dish Problem. Eleanor made a deal with her mom. When she opened a bag of candy to put in her mom's four candy dishes, she would put an equal amount in each dish and then get to eat the ones left over. Last year, her mother received three new candy dishes as presents, and Eleanor believes that she has been getting to eat more leftover candy this year. Could that be true? Explain why or why not.

36. Consider an equation that involves both multiplication and division of whole numbers: Answers vary.

$$a \times b \div c = q \text{ (with remainder } 0 \leq r < c).$$

   a. Taking into consideration the use of the division algorithm to allow a remainder, make predictions about what happens to q and r when a increases, when a decreases, when c increases or decreases, when a increases and c decreases, and so on.
   b. Test your predictions using a spreadsheet or graphing calculator.
   c. State your conclusions in statements such as, If a and b both increase, then q . . . .
   d. Arrange your conclusions into two categories: q decreases if . . .; and q increases if . . . .
   e. Share your categories with other students and look for patterns.

37. Which of the following sets of whole numbers has (have) the closure property for multiplication?
   a. {1, 3, 6}
   b. {0,1}
   c. {10, 20, 30, 40, . . .}
   d. {0}
   e. the set of even whole numbers
   f. the set of odd whole numbers

38. True or false? No finite subset of whole numbers has the closure property for multiplication. Support your answer.

39. Why can't 0 be used as a divisor? Use the definition of division to explain why the following statements are true:
   a. $0 \div a = 0$.
   b. $a \div 0$ is undefined.
   c. $0 \div 0$ is undefined.

40. Return to the picture you drew in Mini-Investigation 2.7 to show the similarities and differences between multiplication and division. What changes would you make to this picture now? Why? Answers vary.

D. Communicating and Connecting Ideas

41. The traditional sequence for teaching operations in the early grades is addition first, followed by subtraction, multiplication, and division. Some early arithmetic texts addressed the four operations in the order of addition, multiplication, subtraction, and division.

With a team of classmates, present a debate, in written or oral form, presenting reasons for and against each approach. Answers vary.

42. **Historical Pathways.** Various definitions have been proposed throughout history for whole-number multiplication and division. An early (1677) definition of multiplication stated, "Multiplication is performed by two numbers of like kind, for the production of a third, which shall have such reason [ratio] to the one, as the other hath to unite [unity, or the number 1]" [Smith (1953), p. 103]. This definition could be restated as $a \times b = c$ means that $c \div a = b \div 1$. Compare this definition to the definitions of multiplication used in this section. Answers vary.

43. **Making Connections.** For many pairs of whole numbers, a and b, no whole number represents $a \div b$. What types of numbers are needed to bring the closure property to division? What physical situations could these numbers model? How would these numbers be related to the set of whole numbers? How could these numbers be helpful in using division to compare two whole numbers? Fractions; additional responses vary.

Section 2.4 Numeration

Numeration Systems ▪ The Hindu–Arabic Place-Value Numeration System ▪ Other Early Numeration Systems ▪ Comparing Numeration Systems

In this section, we examine symbols used to represent whole numbers, with an emphasis on clarity and efficiency of communication. We consider some early methods of representing whole numbers and compare them with the system for representing whole numbers that is used today.

Essential Understandings for Section 2.4

- For any number, the place of a digit tells how many ones, tens, hundreds, and so forth are represented by that digit. A digit times its place value gives the value of the digit. You can add the value of the digits together to get the value of the number.
- Each place value to the left of another is 10 times greater than the one to the right.
- The structure of the base-ten numeration system produces many numerical patterns.
- Place-value periods (thousands, millions, billions, etc.) are used to read and write numbers.
- Whole numbers can be named in equivalent ways using place value.
- Whole numbers can be compared by analyzing corresponding place values.

Mini-Investigation 2.10 may help broaden your ability to communicate number ideas.

Write a paragraph evaluating the strengths and weaknesses of each of your methods.

MINI-INVESTIGATION 2.10 Communicating

Without using the number symbols 0–9 or the number words zero, one, two, . . . nine, devise three different ways to communicate to someone else the number of people in your class.

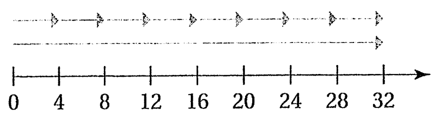
27. **(a)** Associativity of addition and base-ten place value **(b)** One possibility: A pattern in the 9s facts, i.e., $9 + 3 = 12$, $9 + 4 = 13$, etc.
28. **(a)** If a set, S, is closed under an operation, the result of the operation on any element(s) of S is a unique element in S. Each cell in the table contains exactly one whole number. **(b)** The 0 row and 0 column are identical to the row and column representing the other addend. **(c)** Since the top right of the table is the mirror image of the bottom left, $a + b = b + a$.
29. $a < b$ if and only if a is to the left of b on the number line.
30. Answers vary. **(a)** Addition: How many students had lunch at school on Monday? Subtraction: How many more students had lunches from home on Monday than Tuesday? **(b)** $11 + 12 = ?$ $11 - 6 = ?$ Subtraction problems can also be written as $11 = 6 + ?$
32. The application of the mathematical relation $14 - 8 = 6$ to a variety of physical situations demonstrates the generality of mathematics.
33. **(a–e)** If a increases, e increases; if a decreases, e decreases; if c increases, e decreases; if c decreases, e increases; if a increases and c decreases, e increases.
35. False; S = {the odd numbers} is an infinite set not closed under addition.
37. Answers vary. The following table shows one possibility.

#	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

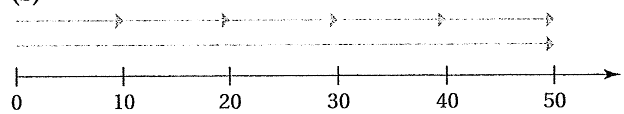
39. **(a)** If the condition $k > 0$ were omitted from the definition of less than, it would be possible for $k = 0$; if so, by the revised definition, $2 < 2$ because there would be a $k = 0$ such that $2 + 0 = 2$.
(b) For a definition of greater than, $a > b$ only if a $k > 0$ exists such that $a = b + k$ or such that $a - k = b$.
40. Answers vary. You could remove objects from a set or move left on a standard number line.
41. Answers vary. Scribes would have to look in the center of the table for the appropriate sum in the row or column of the addend that is being subtracted and then read the other addend from the row or column that forms the intersection at that sum.
42. Answers vary. Some include deficits, losses, temperatures below zero, and elevations below sea level.

SECTION 2.3

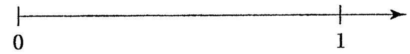
1. **(a)** Show the union of 8 disjoint sets each with four elements. **(b)** Show the union of five disjoint sets each with 10 elements.
2. **(a)**



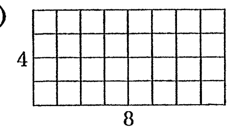
(b)



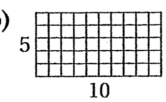
(c) 6 points are coincident



3. **(a)**



(b)



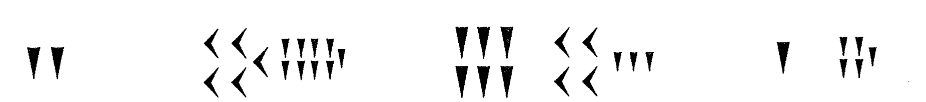
4. If each of 4 children received 30 pieces of candy on Halloween and put all the candy in a bag to share with their classmates, how many pieces of candy are in the bag?
5. If 4 students are running a relay in which each student runs 30 meters, how long is the race?
6. **(a)** $A = 12$ units $\times$ 6 units $= 72$ square units
(b) 42 units $\times$ 1 unit $= 42$ square units
(c) $\frac{b}{2}$ units $\times$ $4b$ units $= \left(\frac{b}{2} \times 4b\right) = 2bb$ square units
7. 4 chairs $\times$ 6 upholstery $=$ 24 covered chairs
8. 3 chocolates $\times$ 5 cheeses $\times$ 3 coffees $=$ 45 baskets
9. w wrapping paper types $\times$ c bow colors $=$ 64 paper-bow combinations
10. **(b)** How many stamps are in a book with 25 pages of 4 stamps each? **(c)** What is the area of a garden 25 feet by 4 feet?
11. **(a)** Commutative property of multiplication **(c)** Distributive property of multiplication over addition **(d)** Closure property of multiplication **(e)** Associative property of multiplication
13. A teacher has 24 geoboards and wants to put them in boxes that will hold 6 each. How many boxes does she need?
14. A shop teacher cut a 24-inch piece of wire into 6-inch segments. How many pieces were made?
15. A teacher has 24 geoboards and wishes to share them evenly between 6 groups of students. How many geoboards will each group get?
16. A shop teacher had a piece of wire 24 inches long. He bent the wire into 6 pieces of equal length. How long was each straight piece?
17. A teacher made 6 tests, each with the same number of questions. The total number of questions was 24. How many questions were on each test?

18. **(a)** Subtract 11 thirteen times **(b)** Subtract 9 twelve times
19. **(a)** $18 = 6 \times n; n = 3$ **(b)** $25 = 5 \times x; x = 5$ **(c)** $y = 42 \times 126 = 5292$ **(d)** $0 = b \times c; b \neq 0$, so $c = 0$
20. **(a)** $15 = n \div 3; 3 = n \div 15$ **(b)** $9 = 9 \div y;$ $y = 9 \div 9$ **(c)** $r = t \div s; s = t \div r$ **(d)** $0 = 0 \div 8$
22. **(a)** **i.** The identity symbol for addition is the zero symbol for multiplication. **ii.** Multiplication and addition are associative, are commutative, have closure, and have an identity property. **iii.** Multiplication has a zero property; addition does not. Multiplication distributes over addition and subtraction; addition does not distribute over other operations. **(b)** Neither subtraction nor division is closed, associative, commutative, has an identity element, or has a zero element. Division does distribute over both addition and subtraction.
23. **(a)** $\{1, 2, 3, 4\} \times \{\,\} = \{\,\}$; no ordered pairs. **(b)** $\{r, s, t\} \times \{a, b\} = \{(r, a), (r, b), (s, a), (s, b), (t, a), (t, b)\}$; 6 ordered pairs **(c)** $\{a\} \times \{5, 6, 7, 8, 9\} = \{(a, 5), (a, 6), (a, 7), (a, 8), (a, 9)\}$; 5 ordered pairs **(d)** $4 \times 0 = 0;$ $3 \times 2 = 6; 1 \times 5 = 5.$
24. No; if $S = \{a, b\}, T = \{a, b\}, S \times T =$ $\{(a, a), (a, b), (b, a), (b, b)\}$. **25.** It is used to generate the numerical value of each digit according to the place it is in; for example, $123 = (1 \times 100) + (2 \times 10) + (3 \times 1)$.
26. **(a)** Closure for multiplication; distributive property of multiplication over addition **(b)** $12 \times 64 = (10 + 2) \times (60 + 4) = (10 \times 60) + (10 \times 4) + (2 \times 60) + (2 \times 4) = 600 + 40 + 120 + 8 = 768.$
28. **(a)** If a set, *S*, is closed under an operation, the result of the operation on any element(s) of *S* is a unique element in *S*. Each cell in the table contains exactly one whole number. **(b)** The 1 row and 1 column are identical to the row and column representing the other factor. **(c)** Since the top right of the table is the mirror image of the bottom left, $a \times b = b \times a$. **(d)** The 0 row and 0 column contain all 0s. **(e)** For example, if you add the 2 column to the 3 column, you get the 5 column.
29. Find the dividend (the product) in the interior of the table in the row or column of the divisor (a factor); then read the quotient (the other factor) from the row or column that forms the intersection at that product.
30. **(a)** The first type of problem asks "How many groups?" and the second asks "How many in one group?" The first type is acted out by removing a "bunch" at a time; the second is dealing out one at a time, etc. **(b)** They each can be thought of as subtracting *d* (the divisor) over and over again. Each represents separating into groups of equal size, etc. **31.** **(a)** The (q, r) pairs are: (0, 64), (1, 52), (2, 40), (3, 28), (4, 16), (5, 4) **(c)** There could be more than one possible quotient and remainder for a given problem.
32. **(a)** $(4 + 6) \div 2; (10 + 15) \div 5; (6 + 9) \div 3;$ $(14 + 49) \div 7$ **(b)** *a* and *b* must be multiples of c because of the restriction of whole number quotients for each division expression. **(c)** The property is valid in the format in **(b)** because of the whole-number quotient restriction.
35. Yes, it could be true, because with 7 candy dishes, the size of the remainder can be greater.
37. All these sets except (a) are closed under multiplication.
38. False, consider {0, 1}. The set is finite and closed under multiplication.
39. **(a)** $0 \div a = 0$ because, for all whole numbers *a*, $0 \times a = 0$. **(b)** $a \div 0$ is undefined because no whole number *b* exists such that $b \times 0 = a$, unless $a = 0$ (see part c). **(c)** $0 \div 0$ is undefined because no unique whole number *b* exists such that $b \times 0 = 0$; every whole number could satisfy this statement.

SECTION 2.4

3. **(a)** 586 = 5 hundreds squares, 8 tens sticks, 6 units cubes **(b)** $4392 = 4(1000) + 3(100) + 9(10) + 2(1)$ **(c)** $2{,}864{,}071 = 2(10^6) + 8(10^5) + 6(10^4) + 4(10^3) + 0(10^2) + 7(10^1) + 1(10^0)$
9. One-to-one correspondence; generation of equivalent sets
14. See answers below.
15. **(a)** Hindu-Arabic has only 10 symbols, but Egyptian has potentially infinitely many; Hindu-Arabic has a symbol for zero, but Egyptian doesn't; Hindu-Arabic uses place value, but Egyptian doesn't; Egyptian uses tallying, but Hindu-Arabic doesn't.

Answer for Exercise 14

	120	59	403	65
(a) Egyptian				
(b) Babylonian				
(c) Roman	CXX	LIX	CDIII	LXV
(d) Mayan