Section 1.1

Communicating Mathematically

Using Multiple Representations to Communicate Mathematically
▪ Using Technology to Communicate Mathematically

In this section, we examine various ways that mathematical ideas can be communicated by using words, symbols, data, and graphs and using technology to display data that can help you explore mathematical ideas and solve problems.

Essential Understandings for Section 1.1

- Mathematical ideas can be communicated in a variety of ways (e.g., through words, graphs, symbols, and objects) including the use of technology.

Talk about the words you used and compare them with those used by other classmates.

MINI-INVESTIGATION 1.1 Communicating

What mathematical words (e.g., square, multiply) would be helpful in giving someone directions for the following activities?

a. Share the cost of a meal among three friends.
b. Find a certain store in a city.
c. Determine the amount of carpet needed to cover a living room floor.
d. Determine an appropriate tip for a waitress.
e. Find a constellation in the night sky.

Using Multiple Representations to Communicate Mathematically

When people think of communication, they typically think about activities such as writing, drawing, speaking, or using body language. The message of this section is that words, symbols, graphs, tables, numerical data, and pictures can be used to represent and communicate ideas. Often these representations are produced and processed using technology. We first look at the role of mathematical words and symbols in communication.

Communicating with Mathematical Words and Symbols. Mathematics is a language of words and symbols that is used to communicate ideas of number, space, and real-world phenomena. For example, the language of mathematics may be used to describe how to get from one store to another by traveling *parallel* or *perpendicular* to certain roads. Or the words *similar* and *congruent* may be used to explain the relationship between pairs of geometric figures.

In both everyday and mathematical situations you have probably used words such as *square*, *diagonal*, and *circle*, which have commonly understood meanings. You may also have stated some relationships, such as *intersect*, *inside*, *on*, or *tangent to*. As you begin to use the language of mathematics meaningfully, you reinforce your understanding of mathematics as you learn to communicate your ideas to others.

Symbols are often used to represent mathematical ideas and can play an important role in communicating those ideas clearly and efficiently. For example, the symbols $E = mc^2$ may stimulate a scientist to think about significant ideas involved in Einstein's theory of relativity. Seeing the symbol π may cause you to think about a relationship involving a circle. Ideas can be represented by symbols, and the symbols can stimulate thinking about those ideas. As you think about familiar mathematical words and symbols and what they mean, you will see that symbols and carefully chosen words can economically represent ideas in mathematics. This use of words and symbols not only allows you to communicate ideas more efficiently, but makes discovering new ideas and important relationships easier. Exercises 11 and 27 in Section 1.1 will help you understand the importance of communication in mathematics.

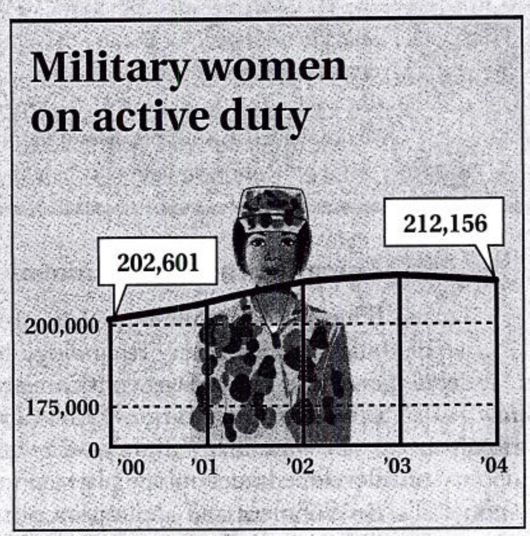
Communicating with Data and Graphs. Data and interpretations of data are used in everyday situations and in newspapers, magazines, and television to represent and communicate information. For example, you might say that the average salary for a certain profession is $45,000 per year or that the median height of students in a class is 5 feet 3 inches. You might also say that the scores on a test ranged from 74 to 98. A newspaper might report a high correlation between cigarette smoking and lung cancer or between exercise and mental health. In the media, data are often displayed in tables, charts, or graphs in order to condense and communicate information effectively.

Graphic artists have become quite sophisticated in their ability to produce just the right graph to communicate what they want the reader to perceive. For example, word processing software often has a feature that allows users to choose a graph and customize it to display the information they want to emphasize.

Write a one-paragraph interpretation of the graph and compare your interpretation with those of others in your class.

MINI-INVESTIGATION 1.2 **Communicating**

What message is communicated by this graph?



Source of data: Department of Defense

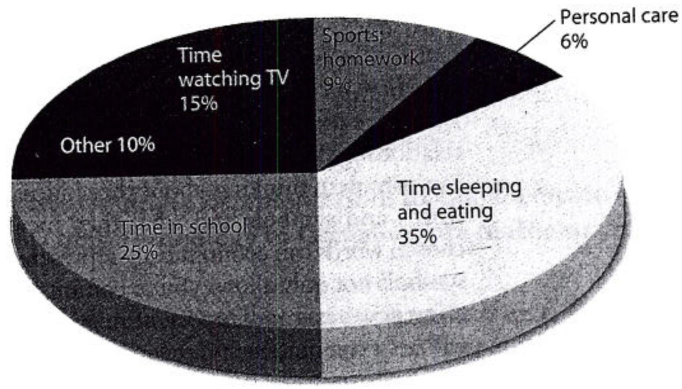


FIGURE 1.1
Circle graph showing the weekday distribution of time for a typical teenager.

Because the clever design of some graphs can disguise or visually alter the meaning or impact of data, everyone needs to be able to interpret correctly the information such graphs convey. We develop several techniques for examining graphs in Chapter 8, but Mini-Investigation 1.2 provides an initial example of how a graph communicates information and allows you to interpret properly what you see.

Another example of how graphs communicate information is shown in Figure 1.1 above. This circle graph visually conveys information about how teenagers spend their time during a typical weekday.

Graphs are an important way of displaying the results of classifying data efficiently and visually. In Chapter 8, we discuss in detail various methods of data analysis and types of graphs used to display data.

Connection to the PreK–8 Classroom

One of the important aspects of doing mathematics in the elementary grades is to encourage students to communicate their mathematical ideas and thinking. The NCTM *Principles and Standards* call for the use of "mathematical communities" in the classroom where the emphasis is on students talking and writing about mathematics. In part, the goal is to encourage the acquisition of the language of mathematics, but, more fundamentally, it is to emphasize the importance of learning that doing mathematics involves communicating and representing mathematical ideas.

Using Technology to Communicate Mathematically

The use of technology plays a central role in representing and communicating mathematical ideas. Technology can be used to show a graph of an equation, to produce and vary a chart containing related data, or to solve and present a solution to a problem. These various forms of representation can be used to communicate ideas.

Because many useful and relatively inexpensive types of technology have been developed, technology plays an increasingly important communication role in the development and application of mathematics. People using mathematics in many different professions now regularly utilize the computer in their daily activities. The computer also helps them communicate with each other to analyze data, share conjectures, and find and present solutions to problems.

SPOTLIGHT ON TECHNOLOGY Computer Spreadsheets

A computer spreadsheet is a grid of rows and columns used to store, organize, and manipulate numbers and other data. The first spreadsheet for a personal computer was VisiCalc. Released in 1979, this software helped popularize desktop computing. Many publishers now offer spreadsheet software for a variety of computers.

The rows and columns of a computer spreadsheet are labeled with numbers and letters, as shown in Figure 1.2. The labels provide an easy way to identify spreadsheet cells. For example, the number 3.8 is in cell B3, the cell where column B and row 3 intersect. Note that spreadsheet cells can contain words or numbers.

The user can enter a formula into a spreadsheet cell to instruct the computer to calculate values from other cells. For example, we could select cell B6 and enter the formula = sum (B3 + B4 + B5). The sum of the three cells in the column above B6 will then be calculated and recorded in cell B6. The value of a computer spreadsheet is obvious when we type in new values in cells B3, B4, and B5. The formula automatically recalculates the sum and replaces the old sum in cell B6.

	A	B	C	D	E
1	Projected Sales by City (millions of dollars)				
2	City	1996	1997	1998	
3	Denver	3.8	4.1	5.1	
4	Miami	7.6	8.9	8.9	
5	Phoenix	4.7	5.1	5.1	
6	Total Sales				
7					
8					

FIGURE 1.2

Portion of a computer spreadsheet showing row and column entries.

Because a computer spreadsheet allows cell values to be changed and formulas used to recalculate solutions, a spreadsheet is an excellent tool for mathematical problem solving. Computer spreadsheets are also being widely used in personal recordkeeping, in small businesses, and in a variety of complex technical applications.

Technology is also being used more and more to help people learn mathematics and to communicate with other learners about mathematics. Initially, educators debated the wisdom of allowing students to use calculators while learning mathematics. The evidence now seems clear that the use of calculators not only does not impede the learning of mathematics but can even enhance students' abilities to solve problems and think conceptually. In the following subsections, we explore three forms of technology that are commonly used in mathematics and that will be referred to throughout this book: computer spreadsheets, calculators, and geometry exploration software.

Using a Computer Spreadsheet. Using a computer spreadsheet for problem solving allows the user to manipulate data and try alternative solutions to the problem quickly and easily. To gain an initial understanding of how a spreadsheet works, let's consider the following problem-solving situation:

> A club is planning a car-wash day to make money. The members think that a charge of \$6 for washing a car is reasonable, but they have found that the cost will be \$25 to rent space for the day and \$2 per car for water and equipment. They now want to know how many cars they would have to wash before they begin to make a profit and how many cars they would have to wash to make a profit of \$100.

To find the answers to these questions, we could use pencil and paper to set up a table. In the first column of the table, we would list the number (n) of cars to be washed. In the next two columns, we would calculate the expense and income for each car washed. We know that the expenses are \$2 for each car washed and \$25 no matter how many are washed. To calculate the expense, we would use the formula Expense $= (2n + 25)$ in the second column of the table. To calculate the income at \$6 per car washed, we would use the use the formula Income $= 6n$ in the third column of the table.

Number of Cars (n)	**Expense $= (2n + 25)$**	**Income $= 6n$**
1	$(2 \times 1) + 25 = 27$	$6 \times 1 = 6$
2	$(2 \times 2) + 25 = 29$	$6 \times 2 = 12$
$\vdots$	$\vdots$	$\vdots$
10	$(2 \times 10) + 25 = 45$	$6 \times 10 = 60$
$\vdots$	$\vdots$	$\vdots$

Alternatively, we could use a computer spreadsheet that finds and displays the answers quickly and easily. We would set up the spreadsheet to display the number of cars in the cells of the first column. Then we would instruct the computer to apply the formulas we used in the table to the cells in the second and third columns. The results are shown in Figure 1.3.

By looking at such a spreadsheet, the club members could easily see that they would begin to make a profit when they had washed the seventh car. If they looked at more rows of the spreadsheet (not shown in Figure 1.3), they would find that they have to wash 32 cars to make a profit of \$100.

	A	B	C
1	Number of Cars	Expense (\$)	Income (\$)
2	1	27	6
3	2	29	12
4	3	31	18
5	4	33	24
6	5	35	30
7	6	37	36
8	7	39	42
9	8	41	48
10	9	43	54
11	10	45	60

FIGURE 1.3

Spreadsheet printout showing part of the expense and income for a proposed car-wash project.

Because different conditions can be easily examined by changing the numbers and the formulas in the spreadsheet, the club members could use the spreadsheet as a decision-making tool. For example, if the club members want to consider raising the price of washing a car to $7, they could see what happens to their expected income by changing the formula and having the computer recalculate the spreadsheet values in the income column. Or they could consider renting less costly space and recalculate the spreadsheet values in the expense column with a revised formula such as Expense $= 2n + 15$. Example 1.1 further illustrates the use of a spreadsheet.

Example 1.1 Problem Solving: Golf Course Fees

A golf course charges a special fee of $14 for each 18-hole round if you agree to play at least five rounds. It also offers an introductory package in which the first five rounds are $7 each, with each round after that costing the regular $18 fee. How many rounds would you have to play for the special fee to be a better deal than the introductory package?

WORKING TOWARD THE SOLUTION

Set up a computer spreadsheet using the data in the example. The cells in the first column should indicate the number of rounds to be played, and the cells in the second and third columns should contain the formulas for calculating the total cost for the number of rounds at the special fee and at the introductory package fee. The values for the first seven rows are shown in the following sample computer spreadsheet.

	A	B	C
1	Rounds	Special Fee ($)	Introductory Package ($)
2	1	14	7
3	2	28	14
4	3	42	21
5	4	56	28
6	5	70	35
7	6	84	53
8	7	98	71

YOUR TURN

Practice: Complete seven more rows of the spreadsheet to solve the problem. Use a paper-and-pencil copy of the spreadsheet and an ordinary calculator to fill in the values. Or use a computer spreadsheet, if a computer and software are available, to find the values.

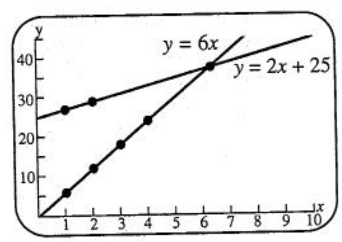
Reflect: Explain the formula you used to find the values for the second and third columns. ■

Using a Calculator. Using a calculator allows routine mathematical tasks to be done quickly and easily. The efficiency that calculators provide frees the user to spend more time thinking about mathematical ideas, patterns, relationships, and problem solving and less time making calculations. Throughout the rest of this book, we provide numerous opportunities for you to try various tasks on different types of calculators. For now, we consider briefly four types of calculators and their unique capabilities.

The simplest calculator is a *four-function calculator* that performs the basic operations of addition, subtraction, multiplication, and division. A second type of calculator is often called a *fraction calculator.* Its primary feature—in addition to the features of a four-function calculator—is its ability to add, subtract, multiply, and divide fractions and mixed numbers. Fraction calculators can also convert an improper fraction to a mixed number, simplify fractions, and convert a fraction to a decimal and vice versa.

A third type of calculator commonly used in mathematics is the *scientific calculator.* In addition to having all the features of a four-function calculator, most scientific calculators have the ability to carry out an entered program to complete a variety of combined calculations automatically. Other features of the scientific calculator include keys for most of the important algebraic and trigonometric functions and capabilities for calculating various statistical measures and producing random numbers.

The fourth type of calculator is the *graphing calculator.* In addition to having all the features of a scientific calculator, a graphing calculator can display graphs of data. For example, we could use the data from the car-wash problem and a graphing calculator to plot graphs of expense and income for various numbers of cars, as shown on the following screen of a graphing calculator.



If we correctly set the scale on the (horizontal) x-axis and the (vertical) y-axis of the graph, the point where the two graphs intersect shows how many cars would have to be washed for the income to equal the expense. Appendix A contains more information on using graphing calculators.

Using Geometry Exploration Software. Another form of technology that can help in discovering and analyzing mathematical ideas is *geometry exploration software.* Appendix B contains information on using geometry exploration software, and in later chapters we explore its use in detail. Let's consider briefly some of the things that geometry exploration software can do.

It allows the user to quickly produce and label a number of different examples of any geometric shape. Some types of this software even allow varying sizes and shapes dynamically in order to investigate the effect of changing them. Geometric exploration software allows immediate measurement of the length of any segment, the size of any angle, and the area of any figure, and it provides the ability to compute with those measurements. The results of these computations are updated automatically when the data are changed.

To illustrate the value of geometry exploration software, suppose that we investigate what happens when we join the midpoints of the sides of a quadrilateral. We can instruct the computer to draw any type of quadrilateral we want, mark the

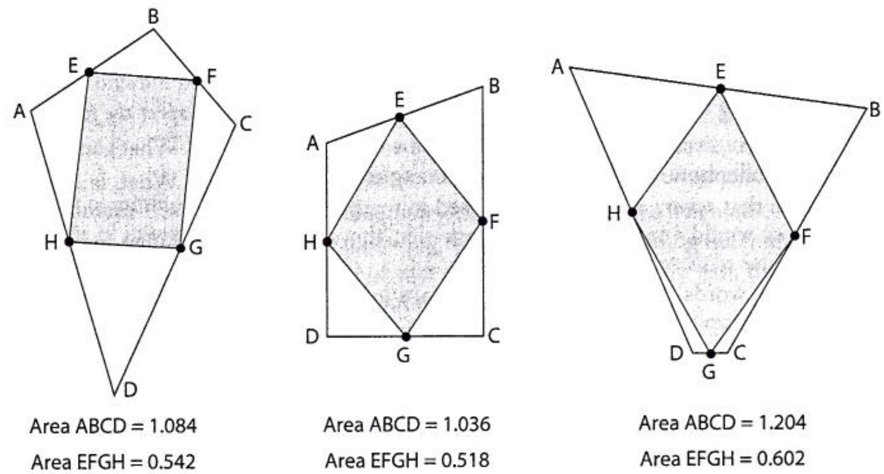


FIGURE 1.4
Quadrilaterals produced by geometry exploration software.

midpoints of its sides, and connect them. We can also instruct the computer to calculate the area of the original large quadrilateral and the area of the smaller inner figure. After we have given these instructions to the computer, it can quickly produce additional random examples of this arrangement. Figure 1.4 shows three geometric figures that the geometry exploration software might produce. Mini-Investigation 1.3 follows up on this approach.

Draw some different quadrilaterals to illustrate your discovery. Then write a statement describing your discovery.

Technology Extension
Use geometry exploration software to produce several more examples of this relationship.

MINI-INVESTIGATION 1.3 Finding a Pattern

From the geometric figures and data shown in Figure 1.4, what can you discover about the following?

a. What geometric figure is formed when the midpoints of the sides of any quadrilateral are connected?

b. How does the area of the original quadrilateral compare to the area of the shaded quadrilateral?

Because geometry exploration software makes producing examples and measuring them so easy, it naturally leads to "what-if" questions. For example, you might be tempted to ask, "What if I were to join the midpoints of the sides of a triangle instead of a quadrilateral?" or "What if I were to join trisection points instead of midpoints?" In this book, especially in Chapters 10 and 11, we use geometry exploration software to explore patterns and relationships in geometric figures.

Connection to the PreK–8 Classroom

The **Technology Principle** in the NCTM *Principles and Standards* emphasizes that technology should be used to help all students learn mathematics. Every student should have access to a calculator and to a computer. Through the use of technology to solve problems and explore mathematical ideas students will be better prepared to live in a technologically oriented world.

Problems and Exercises for Section 1.1

A ☛ icon indicates the answer is at the back of the book.

A. Reinforcing Concepts and Practicing Skills

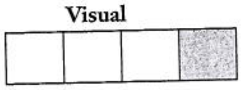
1. Write down directions that you would give a classmate over the telephone for constructing a rectangle. ☛
2. Suppose that *square* was not a word used in mathematics. How would you describe the shape that the word represents? Answers vary.
3. List three words that have special meanings in mathematics and explain their meanings. Answers vary.
4. List three symbols that you have used in mathematics and explain what they represent. ☛
5. List three different ways of representing $\frac{3}{5}$. ☛
6. List three different ways of representing 24. ☛
7. Give a verbal, visual, numerical, and graphic representation for the idea *one-fourth*. Answers vary.

Provide the missing representations in Exercises 8–10.

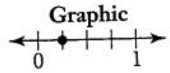
8. **Visual** **Numerical** **Graphic**

$\frac{2}{3}$

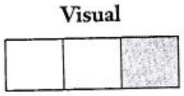
9. **Visual** **Numerical** **Graphic**



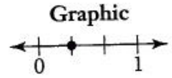
$\frac{1}{4}$



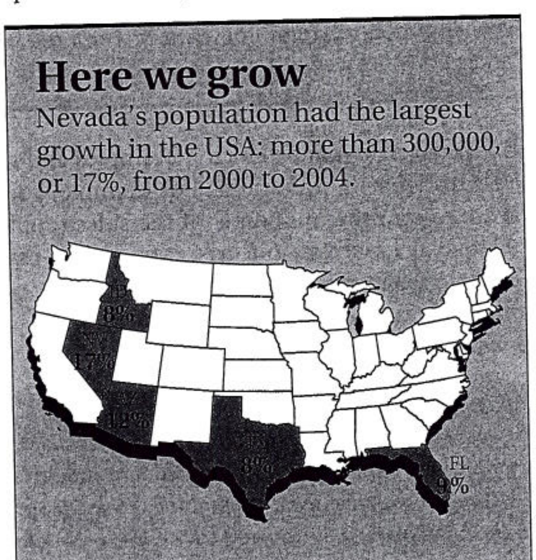
10. **Visual** **Numerical** **Graphic**



$\frac{1}{3}$



11. How would you instruct someone to take one-fourth of a pizza if you could not use the word *one-fourth*? ☛
12. What message is communicated by the following graph? Answers vary.



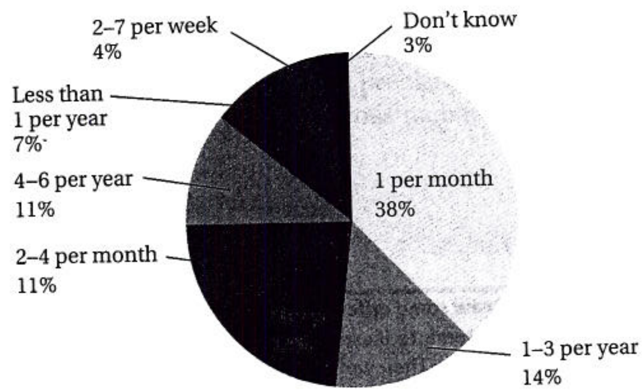
Source of data: U.S. Bureau of the Census

Interpret the following circle graph to answer questions 13–15.

13. What general message does the graph convey to you? ☛
14. What is the most frequent occurrence in terms of self-examinations? one self-examination per month
15. What percentage of women perform self-examinations *at least* once a month? fifty-three percent

REGULAR BREAST SELF-EXAMS

To aid early detection of breast cancer, the No. 2 cause of death for women, women are urged to perform a self-exam monthly. How often women actually perform them:



Source of data: Opinion Research Corp. survey for Sanus Health Plan (1996).

16. Describe another way that the information in the previous circle graph could be represented. Answers vary.
17. Consider the following computer spreadsheet:

	A	B	C
1	Cost of Item ($)	10% Discount ($)	Final Cost ($)
2	80	8	72
3	367	36.70	330.30

What formulas would you use to calculate the numbers in the following cells?

a. B2 B2 = 0.10 * A2

b. C2 C2 = 0.90 * A2

18. Which type of calculator would you use if you wanted to find $\frac{3}{8} + \frac{4}{5}$? fraction calculator
19. Which type of calculator would you use if you wanted to automatically find the value of F for several values of C in the formula $F = (9C \div 5) + 32$? programmable scientific calculator

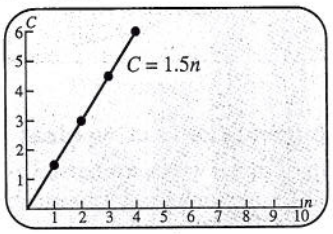
20. Which type of calculator would you use if you wanted to show a graph of the formula in Exercise 19? ☛

21. Which features of geometry exploration software make it useful? Explain why. Answers vary.

B. Deepening Understanding

22. Which of the following do you think best communicates the cost of buying 1, 2, 3, . . . , 10 gallons of gasoline at \$1.50 per gallon? Answers vary.

 a. $C = 1.5n$

 b. 

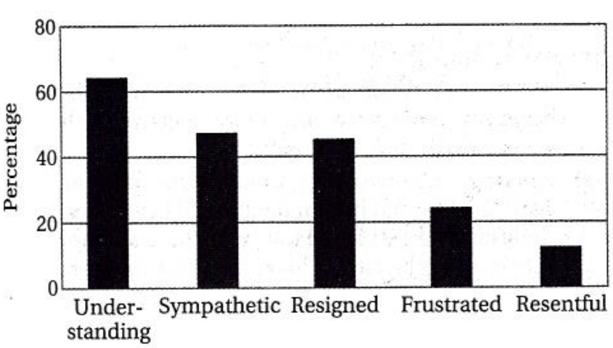
 c.

Number of Gallons	Total Cost (\$)
1	1.50
2	3.00
3	4.50
4	6.00
⋮	⋮

23. Consider the following graph: Answers vary.

COPING WITH MIGRAINES

How migraine sufferers say their families react when headaches keep them out of family activities:



Source of data: Opinion Research Corp. for Glaxo Wellcome, Inc. (1996).

 a. What questions could you answer with this graph?

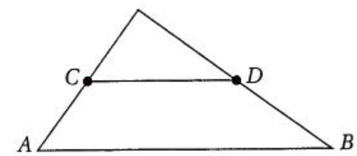
 b. What questions couldn't you answer with this graph?

 c. Do you think that the title of the graph is appropriate? Explain.

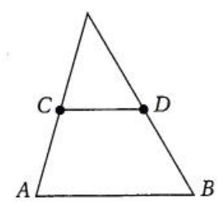
 d. Why is the sum of the percentages greater than 100 percent?

24. Consider the subjects of mathematics, art, and law. Which two do you think are the least alike? Explain. ☛

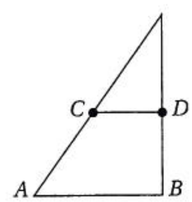
25. Geometry exploration software was used to produce the following three examples of joining the midpoints C and D of two sides of a triangle, along with the accompanying measurements:



Length $CD = 0.840$
Length $AB = 1.680$



Length $CD = 0.520$
Length $AB = 1.040$



Length $CD = 0.460$
Length $AB = 0.920$

What do you discover when you compare the two lengths in each triangle? Write a description of your discovery. ☛

C. Reasoning and Problem Solving

26. **The Car Depreciation Problem.** A newer car costs \$30,000. Each year, the value of the car can be found by multiplying the previous year's value by 0.8. Thus, after the first year the value is \$24,000, after the second year the value is \$19,200, and so on. After what year will the car be valued at less than \$10,000? Describe how you would complete the following spreadsheet to solve the problem. ☛

	A	B
1	Number of Years	Value of Car ($)
2	0	30,000
3	1	
4		
5		
6		

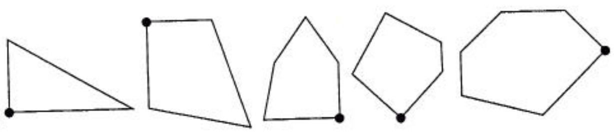
Use the following questions to help you obtain the answer:

a. What formula would you use for cell B3? cell B4? cell B5?

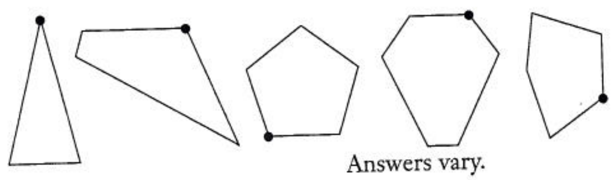
b. How many rows would you have in the spreadsheet? Explain.

27. Describe the idea illustrated in the following display and invent a word that could be used to communicate that idea:

All of these geometric figures illustrate the idea.



None of these geometric figures illustrate the idea.



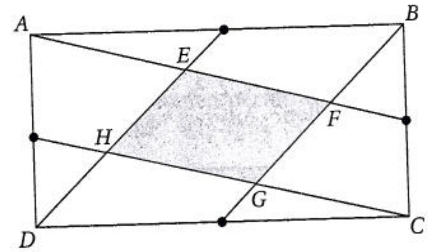
Answers vary.

28. **The Racquet Club Problem.** The following computer spreadsheet shows data for two fee plans for use of a racquet club. Plan A is a flat rate of $18 an hour for play. Plan B involves a beginning fee of $100 and an hourly fee of $8. How many hours would you have to play in order for plan B to be the best plan for you? Use a paper-and-pencil table and a calculator—or use a computer if one is available—to add as many rows as necessary to solve the problem. 11 or more hours

	A	B	C
1	Number of Hours	Plan A ($)	Plan B ($)
2	1	18	108
3	2	36	116
4	3	54	124
5	4	72	132
6	5	90	140

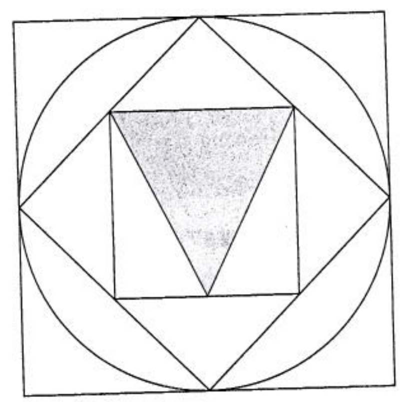
Describe why the spreadsheet can be used effectively to solve the problem.

29. Geometry exploration software was used to produce the following figure (the dots are the midpoints of the sides of *ABCD*) and to measure the area of *ABCD* and *EFGH*: Estimate what part *EFGH* is of *ABCD*. Estimates vary.



D. Communicating and Connecting Ideas

30. Work in a group of three, with each person assigned to one of three roles, to communicate information about the figure shown. The *teller* gives directions on how to draw the figure, without watching the drawer. The *drawer* draws the figure based on the teller's instructions, without looking at the figure. The *analyzer* watches the process, noting mathematical terms used or misused, and comments on the others' efforts.



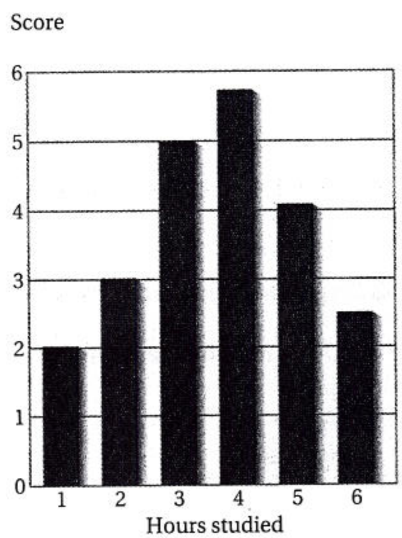
Write your observations about your role, discussing ways you could have been more effective in the communication process. Answers vary.

31. **Making a Connection.** Consider the following problem: A farmer has hens and rabbits. They have a total of 50 heads and 140 feet. How many hens and how many rabbits does the farmer have? 30 hens and 20 rabbits

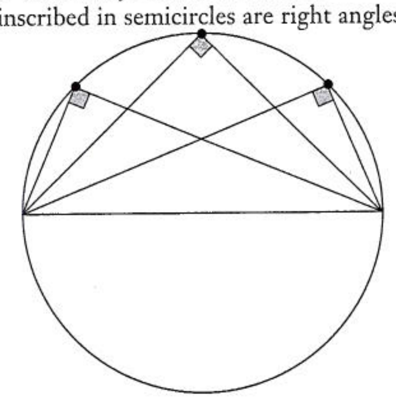
Make a table, or use a computer spreadsheet, to solve the problem. (*Hint:* Let the number of hens range from 50 to 0 and the number of rabbits range from 0 to 50. The number of feet can be determined as twice the number of hens plus four times the number of rabbits.)

32. A researcher determined the amount of time (in hours) that students spent studying for a test, recorded their grades based on a score of 0 (low) to 6 (high), and compiled the graph on the following page.

Write your interpretation of the data shown in the graph and compare your interpretation with those of others in your group. Answers vary.



33. **Historical Pathways.** Sometimes in mathematics a picture can communicate a lot of information. The Greek mathematician Thales (624–547 B.C.) chose points on a semicircle, connected them to the ends of the diameter of the circle, and measured the angle formed each time. He is said to have sacrificed a bull in joy over the discovery communicated by the following diagram. What do you think Thales discovered? Angles inscribed in semicircles are right angles.



Section 1.2

Reasoning Mathematically

Inductive Reasoning and Patterns ▪ Deductive Reasoning

In this section, we examine the view of mathematics as a process of reasoning. We consider two main types of reasoning utilized both in everyday life and in mathematics: inductive reasoning and deductive reasoning.

Essential Understandings for Section 1.2

- One form of mathematical reasoning is inductive reasoning, which involves reaching conclusions based on specific examples.
- Another form of mathematical reasoning is deductive reasoning, which involves reaching conclusions based on rules of logic, such as affirming the hypothesis and denying the conclusion.
- Conditional statements (statements of the form "If. . . , then. . .") and their various forms (negations, converses, inverses, contrapositives, conjunctions, disjunctions, and biconditionals) are an integral part of deductive reasoning.

Reasoning not only plays an important role in making everyday decisions but also plays a central role in mathematics. Doing mathematics often requires the use of several different types of reasoning. For example, someone might use *inductive reasoning* to infer that an odd number multiplied by an even number is an even number. This inference might be based on looking at several examples of odds times evens and observing that in each case the result is an even number. Later the same person