The Greatest Common Factor and Factoring by Grouping

6.1

Recall the following diagram used to illustrate the relationship between multiplication and factoring.

$$\text{Factors} \longrightarrow \underbrace{3 \cdot 5}_{} = 15 \longleftarrow \text{Product}$$

Multiplication

Factoring

A similar relationship holds for multiplication of polynomials. Reading the following diagram from left to right, we say the product of the binomials $x + 2$ and $x + 3$ is the trinomial $x^2 + 5x + 6$. However, if we read in the other direction, we can say that $x^2 + 5x + 6$ factors into the product of $x + 2$ and $x + 3$.

Multiplication

$$\text{Factors} \longrightarrow (x + 2)(x + 3) = x^2 + 5x + 6 \longleftarrow \text{Product}$$

Factoring

In this chapter we develop a systematic method of factoring polynomials.

In this section we will apply the distributive property to polynomials to factor from them what is called the greatest common factor.

(def) DEFINITION *greatest common factor*

The ***greatest common factor*** for a polynomial is the largest monomial that divides (is a factor of) each term of the polynomial.

We use the term *largest monomial* to mean the monomial with the greatest coefficient and highest power of the variable.

EXAMPLE 1 Find the greatest common factor for the polynomial:

$$3x^5 + 12x^2$$

SOLUTION The terms of the polynomial are $3x^5$ and $12x^2$. The largest number that divides the coefficients is 3, and the highest power of x that is a factor of x^5 and x^2 is x^2. Therefore, the greatest common factor for $3x^5 + 12x^2$ is $3x^2$; that is, $3x^2$ is the largest monomial that divides each term of $3x^5 + 12x^2$.

EXAMPLE 2 Find the greatest common factor for:

$$8a^3b^2 + 16a^2b^3 + 20a^3b^3$$

SOLUTION The largest number that divides each of the coefficients is 4. The highest power of the variable that is a factor of a^3b^2, a^2b^3, and a^3b^3 is a^2b^2. The greatest common factor for $8a^3b^2 + 16a^2b^3 + 20a^3b^3$ is $4a^2b^2$. It is the largest monomial that is a factor of each term.

Once we have recognized the greatest common factor of a polynomial, we can apply the distributive property and factor it out of each term. We rewrite the polynomial as the product of its greatest common factor with the polynomial that remains after the greatest common factor has been factored from each term in the original polynomial.

EXAMPLE 3 Factor the greatest common factor from $3x - 15$.

SOLUTION The greatest common factor for the terms $3x$ and 15 is 3. We can rewrite both $3x$ and 15 so that the greatest common factor 3 is showing in each term. It is important to realize that $3x$ means $3 \cdot x$. The 3 and the x are not "stuck" together:

$$3x - 15 = 3 \cdot x - 3 \cdot 5$$

Now, applying the distributive property, we have:

$$3 \cdot x - 3 \cdot 5 = 3(x - 5)$$

To check a factoring problem like this, we can multiply 3 and $x - 5$ to get $3x - 15$, which is what we started with. Factoring is simply a procedure by which we change sums and differences into products. In this case we changed the difference $3x - 15$ into the product $3(x - 5)$. Note, however, that we have not changed the meaning or value of the expression. The expression we end up with is equivalent to the expression we started with. ■

EXAMPLE 4 Factor the greatest common factor from:

$$5x^3 - 15x^2$$

SOLUTION The greatest common factor is $5x^2$. We rewrite the polynomial as:

$$5x^3 - 15x^2 = 5x^2 \cdot x - 5x^2 \cdot 3$$

Then we apply the distributive property to get:

$$5x^2 \cdot x - 5x^2 \cdot 3 = 5x^2(x - 3)$$

To check our work, we simply multiply $5x^2$ and $(x - 3)$ to get $5x^3 - 15x^2$, which is our original polynomial. ■

EXAMPLE 5 Factor the greatest common factor from:

$$16x^5 - 20x^4 + 8x^3$$

SOLUTION The greatest common factor is $4x^3$. We rewrite the polynomial so we can see the greatest common factor $4x^3$ in each term; then we apply the distributive property to factor it out.

$$\begin{aligned} 16x^5 - 20x^4 + 8x^3 &= 4x^3 \cdot 4x^2 - 4x^3 \cdot 5x + 4x^3 \cdot 2 \\ &= 4x^3(4x^2 - 5x + 2) \end{aligned}$$

■

EXAMPLE 6 Factor the greatest common factor from:

$$6x^3y - 18x^2y^2 + 12xy^3$$

SOLUTION The greatest common factor is $6xy$. We rewrite the polynomial in terms of $6xy$ and then apply the distributive property as follows:

$$\begin{aligned} 6x^3y - 18x^2y^2 + 12xy^3 &= 6xy \cdot x^2 - 6xy \cdot 3xy + 6xy \cdot 2y^2 \\ &= 6xy(x^2 - 3xy + 2y^2) \end{aligned}$$

EXAMPLE 7 Factor the greatest common factor from:

$$3a^2b - 6a^3b^2 + 9a^3b^3$$

SOLUTION The greatest common factor is $3a^2b$:

$$\begin{aligned} 3a^2b - 6a^3b^2 + 9a^3b^3 &= 3a^2b(1) - 3a^2b(2ab) + 3a^2b(3ab^2) \\ &= 3a^2b(1 - 2ab + 3ab^2) \end{aligned}$$

Factoring by Grouping

To develop our next method of factoring, called *factoring by grouping*, we start by examining the polynomial $xc + yc$. The greatest common factor for the two terms is c. Factoring c from each term we have:

$$xc + yc = c(x + y)$$

But suppose that c itself was a more complicated expression, such as $a + b$, so that the expression we were trying to factor was $x(a + b) + y(a + b)$, instead of $xc + yc$. The greatest common factor for $x(a + b) + y(a + b)$ is $(a + b)$. Factoring this common factor from each term looks like this:

$$x(a + b) + y(a + b) = (a + b)(x + y)$$

To see how all of this applies to factoring polynomials, consider the polynomial

$$xy + 3x + 2y + 6$$

There is no greatest common factor other than the number 1. However, if we group the terms together two at a time, we can factor an x from the first two terms and a 2 from the last two terms:

$$xy + 3x + 2y + 6 = x(y + 3) + 2(y + 3)$$

The expression on the right can be thought of as having two terms: $x(y + 3)$ and $2(y + 3)$. Each of these expressions contains the common factor $y + 3$, which can be factored out using the distributive property:

$$x(y + 3) + 2(y + 3) = (y + 3)(x + 2)$$

This last expression is in factored form. The process we used to obtain it is called factoring by grouping. Here are some additional examples.

EXAMPLE 8 Factor $ax + bx + ay + by$.

SOLUTION We begin by factoring x from the first two terms and y from the last two terms:

$$\begin{aligned} ax + bx + ay + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y) \end{aligned}$$

To convince yourself that this is factored correctly, multiply the two factors $(a + b)$ and $(x + y)$.

EXAMPLE 9 Factor by grouping: $3ax - 2a + 15x - 10$.

SOLUTION First, we factor a from the first two terms and 5 from the last two terms. Then, we factor $3x - 2$ from the remaining two expressions:

$$\begin{aligned} 3ax - 2a + 15x - 10 &= a(3x - 2) + 5(3x - 2) \\ &= (3x - 2)(a + 5) \end{aligned}$$

Again, multiplying $(3x - 2)$ and $(a + 5)$ will convince you that these are the correct factors. ■

EXAMPLE 10 Factor $2x^2 + 5ax - 2xy - 5ay$.

SOLUTION From the first two terms we factor x. From the second two terms we must factor $-y$ so that the binomial that remains after we do so matches the binomial produced by the first two terms:

$$\begin{aligned} 2x^2 + 5ax - 2xy - 5ay &= x(2x + 5a) - y(2x + 5a) \\ &= (2x + 5a)(x - y) \end{aligned}$$

Another way to accomplish the same result is to use the commutative property to interchange the middle two terms, and then factor by grouping:

$$\begin{aligned} 2x^2 + 5ax - 2xy - 5ay &= 2x^2 - 2xy + 5ax - 5ay && \text{Commutative property} \\ &= 2x(x - y) + 5a(x - y) \\ &= (x - y)(2x + 5a) \end{aligned}$$

This is the same result we obtained previously. ■

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

A. What is the greatest common factor for a polynomial?
B. After factoring a polynomial, how can you check your result?
C. When would you try to factor by grouping?
D. What is the relationship between multiplication and factoring?

Problem Set 6.1

Factor the following by taking out the greatest common factor.

1. $15x + 25$
2. $14x + 21$
3. $6a + 9$
4. $8a + 10$
5. $4x - 8y$
6. $9x - 12y$
7. $3x^2 - 6x - 9$
8. $2x^2 + 6x + 4$
9. $3a^2 - 3a - 60$
10. $2a^2 - 18a + 28$
11. $24y^2 - 52y + 24$
12. $18y^2 + 48y + 32$
13. $9x^2 - 8x^3$
14. $7x^3 - 4x^2$
15. $13a^2 - 26a^3$
16. $5a^2 - 10a^3$
17. $21x^2y - 28xy^2$
18. $30xy^2 - 25x^2y$
19. $22a^2b^2 - 11ab^2$
20. $15x^3 - 25x^2 + 30x$
21. $7x^3 + 21x^2 - 28x$
22. $16x^4 - 20x^2 - 16x$
23. $121y^4 - 11x^4$
24. $25a^4 - 5b^4$
25. $100x^4 - 50x^3 + 25x^2$
26. $36x^5 + 72x^3 - 81x^2$
27. $8a^2 + 16b^2 + 32c^2$
28. $9a^2 - 18b^2 - 27c^2$
29. $4a^2b - 16ab^2 + 32a^2b^2$
30. $5ab^2 + 10a^2b^2 + 15a^2b$
31. $121a^3b^2 - 22a^2b^3 + 33a^3b^3$
32. $20a^4b^3 - 18a^3b^4 + 22a^4b^4$
33. $12x^2y^3 - 72x^5y^3 - 36x^4y^4$
34. $49xy - 21x^2y^2 + 35x^3y^3$

Factor by grouping.

35. $xy + 5x + 3y + 15$
36. $xy + 2x + 4y + 8$
37. $xy + 6x + 2y + 12$
38. $xy + 2y + 6x + 12$
39. $ab + 7a - 3b - 21$
40. $ab + 3b - 7a - 21$
41. $ax - bx + ay - by$
42. $ax - ay + bx - by$
43. $2ax + 6x - 5a - 15$
44. $3ax + 21x - a - 7$
45. $3xb - 4b - 6x + 8$
46. $3xb - 4b - 15x + 20$
47. $x^2 + ax + 2x + 2a$
48. $x^2 + ax + 3x + 3a$
49. $x^2 - ax - bx + ab$
50. $x^2 + ax - bx - ab$

Factor by grouping. You can group the terms together two at a time or three at a time. Either way will produce the same result.

51. $ax + ay + bx + by + cx + cy$
52. $ax + bx + cx + ay + by + cy$

Factor the following polynomials by grouping the terms together two at a time.

53. $6x^2 + 9x + 4x + 6$
54. $6x^2 - 9x - 4x + 6$
55. $20x^2 - 2x + 50x - 5$
56. $20x^2 + 25x + 4x + 5$
57. $20x^2 + 4x + 25x + 5$
58. $20x^2 + 4x - 25x - 5$
59. $x^3 + 2x^2 + 3x + 6$
60. $x^3 - 5x^2 - 4x + 20$
61. $6x^3 - 4x^2 + 15x - 10$
62. $8x^3 - 12x^2 + 14x - 21$

63. The greatest common factor of the binomial $3x + 6$ is 3. The greatest common factor of the binomial $2x + 4$ is 2. What is the greatest common factor of their product $(3x + 6)(2x + 4)$ when it has been multiplied out?

64. The greatest common factors of the binomials $4x + 2$ and $5x + 10$ are 2 and 5, respectively. What is the greatest common factor of their product $(4x + 2)(5x + 10)$ when it has been multiplied out?

65. The following factorization is incorrect. Find the mistake, and correct the right-hand side:

$$12x^2 + 6x + 3 = 3(4x^2 + 2x)$$

66. Find the mistake in the following factorization, and then rewrite the right-hand side correctly:

$$10x^2 + 2x + 6 = 2(5x^2 + 3)$$

Applying the Concepts

67. Investing If you invest \$1,000 in an account with an annual interest rate of r compounded annually, the amount of money you have in the account after one year is:

$$A = 1{,}000 + 1{,}000r$$

Write this formula again with the right side in factored form. Then, find the amount of money in this account at the end of one year if the interest rate is 12%.

68. Investing If you invest P dollars in an account with an annual interest rate of 8% compounded annually, then the amount of money in that account after one year is given by the formula:

$$A = P + 0.08P$$

Rewrite this formula with the right side in factored form, and then find the amount of money in the account at the end of one year if \$500 was the initial investment.

69. Biological Growth If 1,000,000 bacteria are placed in a petri dish and the bacteria have a growth rate of r (a percent expressed as a decimal) per hour, then 1 hour later the amount of bacteria will be $A = 1{,}000{,}000 + 1{,}000{,}000r$ bacteria.

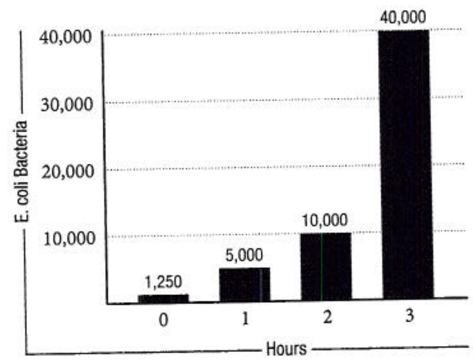
a. Factor the right side of the equation.

b. If $r = 30\%$, find the number of bacteria present after one hour.

70. Biological Growth If there are B E. coli bacteria present initially in a petri dish and their growth rate is r (a percent expressed as a decimal) per hour, then after one hour there will be $A = B + Br$ bacteria present.

a. Factor the right side of this equation.

b. The following bar graph shows the number of E. coli bacteria present initially and the number of bacteria present hours later. Use the bar chart to find B and A in the preceding equation.



Getting Ready for the Next Section

Multiply.

71. $(x - 7)(x + 2)$

72. $(x - 7)(x - 2)$

73. $(x - 3)(x + 2)$

74. $(x + 3)(x - 2)$

75. $(x + 3)(x^2 - 3x + 9)$

76. $(x - 2)(x^2 + 2x + 4)$

77. $(2x + 1)(x^2 + 4x - 3)$

78. $(3x + 2)(x^2 - 2x - 4)$

79. $3x^4(6x^3 - 4x^2 + 2x)$

80. $2x^4(5x^3 + 4x^2 - 3x)$

81. $\left(x + \frac{1}{3}\right)\left(x + \frac{2}{3}\right)$

82. $\left(x + \frac{1}{4}\right)\left(x + \frac{3}{4}\right)$

83. $(6x + 4y)(2x - 3y)$

84. $(8a - 3b)(4a - 5b)$

85. $(9a + 1)(9a - 1)$

86. $(7b + 1)(7b + 1)$

87. $(x - 9)(x - 9)$

88. $(x - 8)(x - 8)$

89. $(x + 2)(x^2 - 2x + 4)$

90. $(x - 3)(x^2 + 3x + 9)$

Factoring Trinomials 6.2

> *Note:* As you will see as we progress through the book, factoring is a tool that is used in solving a number of problems. Before seeing how it is used, however, we first must learn how to do it. So, in this section and the two sections that follow, we will be developing our factoring skills.

In this section we will factor trinomials in which the coefficient of the squared term is 1. The more familiar we are with multiplication of binomials the easier factoring trinomials will be.

Recall multiplication of binomials from Chapter 5:

$$(x + 3)(x + 4) = x^2 + 7x + 12$$
$$(x - 5)(x + 2) = x^2 - 3x - 10$$

The first term in the answer is the product of the first terms in each binomial. The last term in the answer is the product of the last terms in each binomial. The middle term in the answer comes from adding the product of the outside terms to the product of the inside terms.

Let's have a and b represent real numbers and look at the product of $(x + a)$ and $(x + b)$:

$$\begin{aligned}(x + a)(x + b) &= x^2 + ax + bx + ab \\ &= x^2 + (a + b)x + ab\end{aligned}$$

The coefficient of the middle term is the sum of a and b. The last term is the product of a and b. Writing this as a factoring problem, we have:

$$x^2 + \underbrace{(a + b)}_{\text{Sum}}x + \underbrace{ab}_{\text{Product}} = (x + a)(x + b)$$

To factor a trinomial in which the coefficient of x^2 is 1, we need only find the numbers a and b whose sum is the coefficient of the middle term and whose product is the constant term (last term).

EXAMPLE 1 Factor $x^2 + 8x + 12$.

SOLUTION The coefficient of x^2 is 1. We need two numbers whose sum is 8 and whose product is 12. The numbers are 6 and 2:

$$x^2 + 8x + 12 = (x + 6)(x + 2)$$

We can easily check our work by multiplying $(x + 6)$ and $(x + 2)$

$$\begin{aligned}\text{Check:}\quad (x + 6)(x + 2) &= x^2 + 6x + 2x + 12 \\ &= x^2 + 8x + 12\end{aligned}$$

■

EXAMPLE 2 Factor $x^2 - 2x - 15$.

SOLUTION The coefficient of x^2 is again 1. We need to find a pair of numbers whose sum is -2 and whose product is -15. Here are all the possibilities for products that are -15.

Products	Sums
$-1(15) = -15$	$-1 + 15 = 14$
$1(-15) = -15$	$1 + (-15) = -14$
$-5(3) = -15$	$-5 + 3 = -2$
$5(-3) = -15$	$5 + (-3) = 2$

> *Note:* Again, we can check our results by multiplying our factors to see if their product is the original polynomial.

The third line gives us what we want. The factors of $x^2 - 2x - 15$ are $(x - 5)$ and $(x + 3)$:

$$x^2 - 2x - 15 = (x - 5)(x + 3)$$

■

EXAMPLE 3 Factor $2x^2 + 10x - 28$.

SOLUTION The coefficient of x^2 is 2. We begin by factoring out the greatest common factor, which is 2:

$$2x^2 + 10x - 28 = 2(x^2 + 5x - 14)$$

Now, we factor the remaining trinomial by finding a pair of numbers whose sum is 5 and whose product is -14. Here are the possibilities:

Products	Sums
$-1(14) = -14$	$-1 + 14 = 13$
$1(-14) = -14$	$1 + (-14) = -13$
$-7(2) = -14$	$-7 + 2 = -5$
$7(-2) = -14$	$7 + (-2) = 5$

Note: In Example 3 we began by factoring out the greatest common factor. The first step in factoring any trinomial is to look for the greatest common factor. If the trinomial in question has a greatest common factor other than 1, we factor it out first and then try to factor the trinomial that remains.

From the last line we see that the factors of $x^2 + 5x - 14$ are $(x + 7)$ and $(x - 2)$. Here is the complete problem:

$$\begin{aligned} 2x^2 + 10x - 28 &= 2(x^2 + 5x - 14) \\ &= 2(x + 7)(x - 2) \end{aligned}$$

EXAMPLE 4 Factor $3x^3 - 3x^2 - 18x$.

SOLUTION We begin by factoring out the greatest common factor, which is $3x$. Then we factor the remaining trinomial. Without showing the table of products and sums as we did in Examples 2 and 3, here is the complete problem:

$$\begin{aligned} 3x^3 - 3x^2 - 18x &= 3x(x^2 - x - 6) \\ &= 3x(x - 3)(x + 2) \end{aligned}$$

■

EXAMPLE 5 Factor $x^2 + 8xy + 12y^2$.

SOLUTION This time we need two expressions whose product is $12y^2$ and whose sum is $8y$. The two expressions are $6y$ and $2y$ (see Example 1 in this section):

$$x^2 + 8xy + 12y^2 = (x + 6y)(x + 2y)$$

You should convince yourself that these factors are correct by finding their product. ■

Note: Trinomials in which the coefficient of the second-degree term is 1 are the easiest to factor. Success in factoring any type of polynomial is directly related to the amount of time spent working the problems. The more we practice, the more accomplished we become at factoring.

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

A. When the leading coefficient of a trinomial is 1, what is the relationship between the other two coefficients and the factors of the trinomial?
B. When factoring polynomials, what should you look for first?
C. How can you check to see that you have factored a trinomial correctly?
D. Describe how you would find the factors of $x^2 + 8x + 12$.

Problem Set 6.2

Factor the following trinomials.

1. $x^2 + 7x + 12$
2. $x^2 + 7x + 10$
3. $x^2 + 3x + 2$
4. $x^2 + 7x + 6$
5. $a^2 + 10a + 21$
6. $a^2 - 7a + 12$
7. $x^2 - 7x + 10$
8. $x^2 - 3x + 2$
9. $y^2 - 10y + 21$
10. $y^2 - 7y + 6$
11. $x^2 - x - 12$
12. $x^2 - 4x - 5$
13. $y^2 + y - 12$
14. $y^2 + 3y - 18$
15. $x^2 + 5x - 14$
16. $x^2 - 5x - 24$
17. $r^2 - 8r - 9$
18. $r^2 - r - 2$
19. $x^2 - x - 30$
20. $x^2 + 8x + 12$
21. $a^2 + 15a + 56$
22. $a^2 - 9a + 20$
23. $y^2 - y - 42$
24. $y^2 + y - 42$
25. $x^2 + 13x + 42$
26. $x^2 - 13x + 42$

Factor the following problems completely. First, factor out the greatest common factor, and then factor the remaining trinomial.

27. $2x^2 + 6x + 4$
28. $3x^2 - 6x - 9$
29. $3a^2 - 3a - 60$
30. $2a^2 - 18a + 28$
31. $100x^2 - 500x + 600$
32. $100x^2 - 900x + 2{,}000$
33. $100p^2 - 1{,}300p + 4{,}000$
34. $100p^2 - 1{,}200p + 3{,}200$
35. $x^4 - x^3 - 12x^2$
36. $x^4 - 11x^3 + 24x^2$
37. $2r^3 + 4r^2 - 30r$
38. $5r^3 + 45r^2 + 100r$
39. $2y^4 - 6y^3 - 8y^2$
40. $3r^3 - 3r^2 - 6r$
41. $x^5 + 4x^4 + 4x^3$
42. $x^5 + 13x^4 + 42x^3$
43. $3y^4 - 12y^3 - 15y^2$
44. $5y^4 - 10y^3 + 5y^2$
45. $4x^4 - 52x^3 + 144x^2$
46. $3x^3 - 3x^2 - 18x$

Factor the following trinomials.

47. $x^2 + 5xy + 6y^2$
48. $x^2 - 5xy + 6y^2$
49. $x^2 - 9xy + 20y^2$
50. $x^2 + 9xy + 20y^2$
51. $a^2 + 2ab - 8b^2$
52. $a^2 - 2ab - 8b^2$
53. $a^2 - 10ab + 25b^2$
54. $a^2 + 6ab + 9b^2$
55. $a^2 + 10ab + 25b^2$
56. $a^2 - 6ab + 9b^2$
57. $x^2 + 2xa - 48a^2$
58. $x^2 - 3xa - 10a^2$
59. $x^2 - 5xb - 36b^2$
60. $x^2 - 13xb + 36b^2$

Factor completely.

61. $x^4 - 5x^2 + 6$

62. $x^6 - 2x^3 - 15$

63. $x^2 - 80x - 2{,}000$

64. $x^2 - 190x - 2{,}000$

65. $x^2 - x + \frac{1}{4}$

66. $x^2 - \frac{2}{3}x + \frac{1}{9}$

67. $x^2 + 0.6x + 0.08$

68. $x^2 + 0.8x + 0.15$

69. If one of the factors of $x^2 + 24x + 128$ is $x + 8$, what is the other factor?

70. If one factor of $x^2 + 260x + 2{,}500$ is $x + 10$, what is the other factor?

71. What polynomial, when factored, gives $(4x + 3)(x - 1)$?

72. What polynomial factors to $(4x - 3)(x + 1)$?

Getting Ready for the Next Section

Multiply using the FOIL method.

73. $(6a + 1)(a + 2)$

74. $(6a - 1)(a - 2)$

75. $(3a + 2)(2a + 1)$

76. $(3a - 2)(2a - 1)$

77. $(6a + 2)(a + 1)$

78. $(3a + 1)(2a + 2)$

More Trinomials to Factor 6.3

We will now consider trinomials whose greatest common factor is 1 and whose leading coefficient (the coefficient of the squared term) is a number other than 1.

Suppose we want to factor the trinomial $2x^2 - 5x - 3$. We know the factors (if they exist) will be a pair of binomials. The product of their first terms is $2x^2$ and the product of their last term is -3. Let us list all the possible factors along with the trinomial that would result if we were to multiply them together. Remember, the middle term comes from the product of the inside terms plus the product of the outside terms.

Binomial Factors	First Term	Middle Term	Last Term
$(2x - 3)(x + 1)$	$2x^2$	$-x$	-3
$(2x + 3)(x - 1)$	$2x^2$	$+x$	-3
$(2x - 1)(x + 3)$	$2x^2$	$+5x$	-3
$(2x + 1)(x - 3)$	$2x^2$	$-5x$	-3

We can see from the last line that the factors of $2x^2 - 5x - 3$ are $(2x + 1)$ and $(x - 3)$. There is no straightforward way, as there was in the previous section, to find the factors, other than by trial and error or by simply listing all the possibilities. We look for possible factors that, when multiplied, will give the correct first and last terms, and then we see if we can adjust them to give the correct middle term.

EXAMPLE 1 Factor $6a^2 + 7a + 2$.

SOLUTION We list all the possible pairs of factors that, when multiplied together, give a trinomial whose first term is $6a^2$ and whose last term is $+2$.

Binomial Factors	First Term	Middle Term	Last Term
$(6a + 1)(a + 2)$	$6a^2$	$+13a$	$+2$
$(6a - 1)(a - 2)$	$6a^2$	$-13a$	$+2$
$(3a + 2)(2a + 1)$	$6a^2$	$+7a$	$+2$
$(3a - 2)(2a - 1)$	$6a^2$	$-7a$	$+2$

Note: Remember, we can always check our results by multiplying the factors we have and comparing that product with our original polynomial.

The factors of $6a^2 + 7a + 2$ are $(3a + 2)$ and $(2a + 1)$.

Check: $(3a + 2)(2a + 1) = 6a^2 + 7a + 2$ ■

Notice that in the preceding list we did not include the factors $(6a + 2)$ and $(a + 1)$. We do not need to try these since the first factor has a 2 common to each term and so could be factored again, giving $2(3a + 1)(a + 1)$. Since our original trinomial, $6a^2 + 7a + 2$, did *not* have a greatest common factor of 2, neither of its factors will.

EXAMPLE 2 Factor $4x^2 - x - 3$.

SOLUTION We list all the possible factors that, when multiplied, give a trinomial whose first term is $4x^2$ and whose last term is -3.

Binomial Factors	First Term	Middle Term	Last Term
$(4x + 1)(x - 3)$	$4x^2$	$-11x$	-3
$(4x - 1)(x + 3)$	$4x^2$	$+11x$	-3
$(4x + 3)(x - 1)$	$4x^2$	$-x$	-3
$(4x - 3)(x + 1)$	$4x^2$	$+x$	-3
$(2x + 1)(2x - 3)$	$4x^2$	$-4x$	-3
$(2x - 1)(2x + 3)$	$4x^2$	$+4x$	-3

The third line shows that the factors are $(4x + 3)$ and $(x - 1)$.

$$\text{Check:} \quad (4x + 3)(x - 1) = 4x^2 - x - 3$$

You will find that the more practice you have at factoring this type of trinomial, the faster you will get the correct factors. You will pick up some shortcuts along the way, or you may come across a system of eliminating some factors as possibilities. Whatever works best for you is the method you should use. Factoring is a very important tool, and you must be good at it.

EXAMPLE 3 Factor $12y^3 + 10y^2 - 12y$.

SOLUTION We begin by factoring out the greatest common factor, $2y$:

$$12y^3 + 10y^2 - 12y = 2y(6y^2 + 5y - 6)$$

Note: Once again, the first step in any factoring problem is to factor out the greatest common factor if it is other than 1.

We now list all possible factors of a trinomial with the first term $6y^2$ and last term -6, along with the associated middle terms.

Possible Factors	Middle Term When Multiplied
$(3y + 2)(2y - 3)$	$-5y$
$(3y - 2)(2y + 3)$	$+5y$
$(6y + 1)(y - 6)$	$-35y$
$(6y - 1)(y + 6)$	$+35y$

The second line gives the correct factors. The complete problem is:

$$\begin{aligned} 12y^3 + 10y^2 - 12y &= 2y(6y^2 + 5y - 6) \\ &= 2y(3y - 2)(2y + 3) \end{aligned}$$

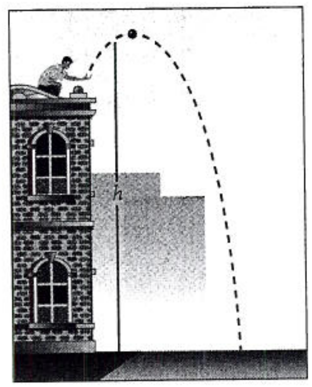
EXAMPLE 4 Factor $30x^2y - 5xy^2 - 10y^3$.

SOLUTION The greatest common factor is $5y$:

$$\begin{aligned} 30x^2y - 5xy^2 - 10y^3 &= 5y(6x^2 - xy - 2y^2) \\ &= 5y(2x + y)(3x - 2y) \end{aligned}$$

EXAMPLE 5 A ball is tossed into the air with an upward velocity of 16 feet per second from the top of a building 32 feet high. The equation that gives the height of the ball above the ground at any time t is

$$h = 32 + 16t - 16t^2$$



Factor the right side of this equation and then find h when t is 2.

SOLUTION We begin by factoring out the greatest common factor, 16. Then, we factor the trinomial that remains:

$$h = 32 + 16t - 16t^2$$

$$h = 16(2 + t - t^2)$$

$$h = 16(2 - t)(1 + t) \qquad \text{Letting } t = 2 \text{ in the equation, we have}$$

$$h = 16(0)(3) = 0$$

When t is 2, h is 0. ■

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

A. What is the first step in factoring a trinomial?
B. Describe the criteria you would use to set up a table of possible factors of a trinomial.
C. What does it mean if you factor a trinomial and one of your factors has a greatest common factor of 3?
D. Describe how you would look for possible factors of $6a^2 + 7a + 2$.

Problem Set 6.3

Factor the following trinomials.

1. $2x^2 + 7x + 3$
2. $2x^2 + 5x + 3$
3. $2a^2 - a - 3$
4. $2a^2 + a - 3$
5. $3x^2 + 2x - 5$
6. $3x^2 - 2x - 5$
7. $3y^2 - 14y - 5$
8. $3y^2 + 14y - 5$
9. $6x^2 + 13x + 6$
10. $6x^2 - 13x + 6$
11. $4x^2 - 12xy + 9y^2$
12. $4x^2 + 12xy + 9y^2$
13. $4y^2 - 11y - 3$
14. $4y^2 + y - 3$
15. $20x^2 - 41x + 20$
16. $20x^2 + 9x - 20$
17. $20a^2 + 48ab - 5b^2$
18. $20a^2 + 29ab + 5b^2$
19. $20x^2 - 21x - 5$
20. $20x^2 - 48x - 5$
21. $12m^2 + 16m - 3$
22. $12m^2 + 20m + 3$
23. $20x^2 + 37x + 15$
24. $20x^2 + 13x - 15$
25. $12a^2 - 25ab + 12b^2$
26. $12a^2 + 7ab - 12b^2$
27. $3x^2 - xy - 14y^2$
28. $3x^2 + 19xy - 14y^2$
29. $14x^2 + 29x - 15$
30. $14x^2 + 11x - 15$
31. $6x^2 - 43x + 55$
32. $6x^2 - 7x - 55$
33. $15t^2 - 67t + 38$
34. $15t^2 - 79t - 34$

Factor each of the following completely. Look first for the greatest common factor.

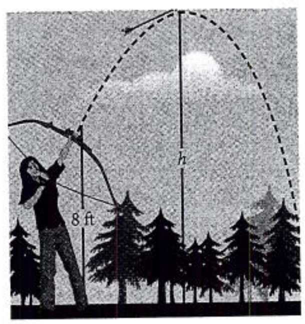
35. $4x^2 + 2x - 6$
36. $6x^2 - 51x + 63$
37. $24a^2 - 50a + 24$
38. $18a^2 + 48a + 32$
39. $10x^3 - 23x^2 + 12x$
40. $10x^4 + 7x^3 - 12x^2$
41. $6x^4 - 11x^3 - 10x^2$
42. $6x^3 + 19x^2 + 10x$
43. $10a^3 - 6a^2 - 4a$
44. $6a^3 + 15a^2 + 9a$
45. $15x^3 - 102x^2 - 21x$
46. $2x^4 - 24x^3 + 64x^2$
47. $35y^3 - 60y^2 - 20y$
48. $14y^4 - 32y^3 + 8y^2$
49. $15a^4 - 2a^3 - a^2$
50. $10a^5 - 17a^4 + 3a^3$
51. $24x^2y - 6xy - 45y$
52. $8x^2y^2 + 26xy^2 + 15y^2$
53. $12x^2y - 34xy^2 + 14y^3$
54. $12x^2y - 46xy^2 + 14y^3$
55. Evaluate the expression $2x^2 + 7x + 3$ and the expression $(2x + 1)(x + 3)$ for $x = 2$.
56. Evaluate the expression $2a^2 - a - 3$ and the expression $(2a - 3)(a + 1)$ for $a = 5$.
57. What polynomial factors to $(2x + 3)(2x - 3)$?
58. What polynomial factors to $(5x + 4)(5x - 4)$?
59. What polynomial factors to $(x + 3)(x - 3)(x^2 + 9)$?
60. What polynomial factors to $(x + 2)(x - 2)(x^2 + 4)$?

Applying the Concepts

61. Archery Margaret shoots an arrow into the air. The equation for the height (in feet) of the tip of the arrow is:

$$h = 8 + 62t - 16t^2$$

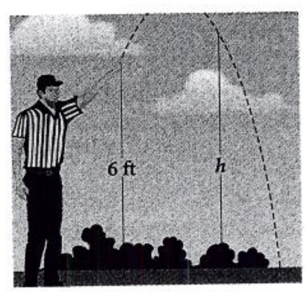
Factor the right side of this equation. Then fill in the table for various heights of the arrow, using the factored form of the equation.



Time t (seconds)	0	1	2	3	4
Height h (feet)					

62. Coin Toss At the beginning of every football game, the referee flips a coin to see who will kick off. The equation that gives the height (in feet) of the coin tossed in the air is:

$$h = 6 + 29t - 16t^2$$



a. Factor this equation.

b. Use the factored form of the equation to find the height of the quarter after 0 seconds, 1 second, and 2 seconds.

63. Constructing a Box Yesterday I was experimenting with how to cut and fold a certain piece of cardboard to make a box with different volumes. Unfortunately, today I have lost both the cardboard and most of my notes. I remember that I made the box by cutting equal squares from the corners then folding up the side flaps:

I don't remember how big the cardboard was, and I can only find the last page of notes, which says that if x is the length of a side of a small square (in inches), then the volume is $V = 99x - 40x^2 + 4x^3$.

a. Factor the right side of this expression completely.

b. What were the dimensions of the original piece of cardboard?

64. Constructing a Box Repeat Problem 63 if the remaining formula is $V = 15x - 16x^2 + 4x^3$.

Getting Ready for the Next Section

Multiply each of the following.

65. $(x + 3)(x - 3)$

66. $(x - 4)(x + 4)$

67. $(x + 5)(x - 5)$

68. $(x - 6)(x + 6)$

69. $(x + 7)(x - 7)$

70. $(x - 8)(x + 8)$

71. $(x + 9)(x - 9)$

72. $(x - 10)(x + 10)$

73. $(2x - 3y)(2x + 3y)$

74. $(5x - 6y)(5x + 6y)$

75. $(x^2 + 4)(x + 2)(x - 2)$

76. $(x^2 + 9)(x + 3)(x - 3)$

77. $(x + 3)^2$

78. $(x - 4)^2$

79. $(x + 5)^2$

80. $(x - 6)^2$

81. $(x + 7)^2$

82. $(x - 8)^2$

83. $(x + 9)^2$

84. $(x - 10)^2$

85. $(2x + 3)^2$

86. $(3x - y)^2$

87. $(4x - 2y)^2$

88. $(5x - 6y)^2$