Numbers which can be written as a fraction or ratio of two integers are called the **rational numbers.** Recall that a fraction is the numerator divided by the denominator, so we can write any fraction as a decimal by dividing its top by its bottom. If we do so, one of two things will happen. For the first case, we consider the fraction , which is to say 3 divided by 8 for the first possibility:

.375

8 ) 3.000

-24

60

-56

40

-40

0

In this case we obtain a remainder of zero, which is to say that 8 divides into 3 exactly as .375; This is called a **terminating decimal**, meaning that after a limited number of places (in this case three) the decimal stops. The other case can be represented by the fraction . In this case, if we divide 1 by 7 we get:

.142857

7 ) 1.000000

-7

30

-28

20

-14

60

-56

40

- 35

50

- 49

1

In this case, we end up with a remainder of 1, which is **the same as the original numerator.** We could continue dividing but because we have already divided into one, the same exact sequence of numbers would be repeated indefinitely. As soon as we rounded (no matter after how many places we rounded) the division would no longer be exact, but clearly we could never write this decimal out to its final place either. Fortunately we have a way around this. Because the same exact sequence is repeated, we can write it with a **repeating bar** (or repetenz) written over it. That is to say,

= .142857 meaning the sequence of six digits written over and over and over again.

All rational numbers, if written as a decimal, will either be a repeating or a terminating decimal. To understand why, consider the remainders in the division we just did.

.142857

7 ) 1.000000

-7

***3***0

-28

***2***0

-14

***6***0

-56

***4***0

- 35

***5***0

- 49

***1***

Notice that all six possible remainders when we divide by seven, 1,2,3,4,5,6 show up in this division. So by the end of it we have simply run out of possible remainders. We either get a remainder of zero (in which case the decimal terminates) or a remainder we have seen before (in which case the decimal repeats.) There are no other possibilities. Note that while seven did go through all six possible remainders before it repeated (and that the number of possible remainders forms an upper limit on how many places a division can go through, so that for example can and does repeat after 12 places) it does not have to go through all possible remainders. For example, the fraction repeats after only three places, and is written = .083 . The third decimal is the only one that repeats so the repeating bar is written only over the place that it applies to.

Example 1: Write the fraction as a decimal.

.27

11) 3.000

-22

80

-77

3

We get the remainder of 3, which was the original numerator so the same division would be repeated and we can write

= .27

Another way of expressing a fraction or decimal is as a **percent.** The term, ‘percent’ means ‘per hundred.’ So a percent symbol (%) means a fraction written over a hundred. A straightforward example would be a decimal which ends in the hundredths place, for example .37 which is literally 37 hundredths, and can therefore be written as 37 %. In general though even if the decimal does not end in the hundredths place we can make it into percent by multiplying by 100%. For example,

Example 2: Write 8.4161 as a percent.

8.4161 X 100% = 841.61 %. Be sure to write the unit, %.

We can similarly write a fraction as a percent by multiplying by 100 %. Note that the net effect of this is that if you are changing a decimal to a percent, you will move the decimal place two places to the right and then write the unit %.

NOTE: WHEN YOU MULTIPLY FRACTIONS YOU MUST FIRST WRITE THEM AS IMPROPER FRACTIONS!

Note also that 100 % can be considered as 100%/ 1, so that the number 100 can cancel against the denominator of the fraction.

Example 3: Write as a percent:

X 100 % = = = = 62 %.

Example 4: Write 1 as a percent:

1 = = X 100 % = = = = 183 %

NOTE: ALTHOUGH WE NEEDED IMPROPER FRACTIONS FOR PURPOSES OF COMPUTATION, WE WRITE THE ANSWER WITH MIXED FRACTIONS!

The reason for this is because % is a unit, and as a unit-valued number it is, for example, easier for someone to understand how much of something 62 % is than %.

To get rid of percentages, you do the opposite of creating them, that is to say you divide by 100%. If you have a decimal percentage to begin with, that is relatively easy in that the decimal will move two places back to the left. Also because you are getting rid of percent you will no longer have a unit to write.

Example 5: Write 5.25% as a decimal number.

5.25% ÷ 100% = .0525 (note that we can only move the decimal one place to the left using only the original digits so we have to put a zero in front and move the decimal another place to the left past the zero.

If you want to convert a fractional percentage to a straight fraction then you should write it as an improper fraction (remember that improper fractions are easier to do mathematical computations with) and instead of dividing by 100%, multiply it by the fraction . As we will discuss later, that is the exact same operation but is much easier to do with fractions.

Example 6: Convert 16 % to a fraction.

16 % = = X = X =

**ADDING AND SUBTRACTING RATIONAL NUMBERS:**

Suppose that you order two pizzas. Each of them is cut into six slices. Assume that each slice is the same size. You eat one slice of the first pizza and three slices of the second pizza. How much pizza did you actually eat?

The first step is to determine the fraction of each pizza that you ate. The numerator is the number of slices of the pizza while the denominator is the number of total slices the pizza was divided into. Or conversely the denominator tells you what portion of a whole pizza is represented by each slice.

+ = = = of a pizza.

Note that the number of pieces is added together, or in other words you ate 1 + 3 = 4 pieces of pizza. However the size of each piece remains the same, of a pizza.

When you want to add two fractions together it is important that they have the same denominators. This is because the denominator represents (as it did in this example) a portion of a whole, and just adding more of the pieces together will not change what portion of a whole each piece is. Here, after we added together (but before we reduced) we still had a fraction with a denominator of 6, representing the fact that combining the number of slices you ate of each pizza together has not changed the size of a slice.

Example 7: Add + - = =

Since there is no unit it is not necessary to write this as a mixed fraction, though you could if you wanted to in which case the answer would be 1 .

Now, suppose that you order two pizzas again, but this time the pizzas are sliced differently. The first pizza is still cut into six pieces but the second is cut into eight pieces. As before you eat one piece of the first pizza and three of the second pizza.

+

Now we have different sized pieces. It is not enough just to count how many pieces we’ve eaten to know how much pizza we’ve had. If we want to add these up, we need to add pieces that are the same size.

So how could we make the pieces the same size. Clearly where a pizza has already been cut, we can’t glue pieces back together (and a glue-and-pepperoni pizza would be BLECCHH!) so we have to incorporate existing cuts. So, for example, the six piece pizza could be further cut into twelve pieces if we cut each piece in half. Or it could be cut into an eighteen piece pizza if we cut each piece into three smaller pieces, etc. But to maintain equality of size we have to subdivide each of the original six slices equally. Thus the number of smaller slices we could subdivide the pizza into would have to be a multiple of six. We could leave it as a six slice pizza, or we could subdivide it into a pizza of 12, 18, 24, 30, 36… slices (at least until we reach the point where we get cheesesticks with a little bread on the bottom.)

Similarly when we subdivide the pizza that originally was in eighths, we could cut each piece into two smaller pieces (making a 16 slice pizza) or we could cut into 24, 32, 40,… pieces.

Looking at both of these pizza options, we see that the number 24 is in both lists. We could cut each slice of the six piece pizza into fourths and each slice of the eight piece pizza into thirds and in that case both of them would become 24-slice pizzas. The one piece of the six slice pizza would then become four slices so that we would be able to say that we had eaten of that pizza, and the three slices of the original eight slice pizza would each be cut into three smaller slices for a total of nine, so that we would be able to say that we had eaten of that pizza. We could then add to get the total amount of pizza eaten as:

+ = = of a pizza.

As before, the numerator counts the number of pieces eaten so we add. But the denominator still represents the portion of a whole pizza in each slice. So once we have made both pizzas the same by cutting them into 24 slices, adding slices together does not change the size of a slice so the denominator remains 24.

The number 24 is the **least common multiple** of 6 and 8, in other words the smallest number that 6 and 8 both go into. Since 6 and 8 were in the denominators of the fractions to begin with and the goal was to make them into the same denominator, we call 24 in this context the **least common denominator** (written from now on as ‘LCD’) of 6 and 8.

Could we add and without having to draw pictures of pizzas and do all of this analysis?

We can, once we know the LCD. To find the LCD it is OK to write out multiples of 6 and multiples of 8 to find what comes first, but in math generally we instead look at the prime factorizations of each number. In Algebra II (next semester) we will find that an understanding of this method is important. So we begin by prime factoring both 6 and 8. 6 = 2·3 and 8 = 2·2·2. Putting all of this together we get that the LCD must contain three factors of 2 (any less and 8 would not go into it) and also a factor 3. Hence the LCD is 2·2·2·3 = 24. Looking at the prime factorization of the LCD we can find either 2·2·2 (eight) or 2·3 (six) inside of it.

Once we know the LCD we multiply each fraction by one. As we know multiplying any number by one does not change the number. In particular the form of one we will multiply each fraction by will be a fraction in which the numerator and denominator are equal. The denominator will then be what is missing to complete the LCD when multiplied by the denominator of the original fraction. In English, this means multiply the bottom by whatever number is needed to turn it into the LCD and then multiply the top by the same number.

So in our example, · + · = + = + = .

Example 8: Add + .

The least common multiple of 5 and 10 is 10. This means we will only have to multiply the first fraction, to make it have a denominator of 10.

· + = + = = (don’t forget to reduce your answer.)

Example 9: Add - +

First note that the LCD of (8,12,6) is 24. So we will have to make each denominator into 24.

· - · + · = - + = - + = = .

**MULTIPLYING AND DIVIDING RATIONAL NUMBERS**

The first thing to recognize is that signs work the same way they do for multiplication and division of integers; that is to say that two negative fractions multiplied or divided together result in a positive answer, while one negative and one positive will result in a negative answer.

We’ve already seen a number of examples in which we multiply fractions together by multiplying across the top and across the bottom, both in the percentage examples above and in creating the common denominator when adding and subtracting. Of course this method of multiplication was covered in pre-algebra, but often the hangup comes when larger fractions are multiplied together and not until later are they reduced. In fact, it is often much easier to reduce fractions **before** you multiply. You can reduce across a multiplication symbol (i.e. the numerator of one fraction against the denominator of the other fraction) as long as everything is being multiplied (there are no plus or minus signs in either numerator or denominator.)

Example 10: · = · = = .

Division of fractions means you need to **invert and multiply**; that is to recognize that division by a fraction is the same as multiplying by its reciprocal. Your first fraction is unaffected, but your second fraction is inverted—that means turned upside down, and then multiplied with the first fraction rather than being divided. The sign if there is one is unaffected by inverting and multiplying.

Example 11:

÷ = ·

Note that two negative numbers are positive.

· = · = = .

NOTE: YOU HAVE TO INVERT AND MULTIPLY **BEFORE** YOU REDUCE.

For example in the preceding example, many students may have seen the 15 and the 10 and assumed that you could cancel a factor of 5. But in fact after you invert and multiply the 15 and the 10 both end up in the numerator so that the factors of five, rather than reducing multiply together to give 25.

**EXPONENTS**:

Exponents are powers which tell you how many of something to multiply together. For example, in 23 the exponent is three, meaning that you should multiply the number 2 together 3 times. Therefore,

23 = 2·2·2 = 8.

Example 12: Find 34.

34 = 3·3·3·3 = 81.

If a number is preceded by a sign, the exponent **does not affect the sign unless the number with the sign is inside parentheses and the exponent is outside the parentheses.**

Example 13: Find -52

-52 = -5·5 = - 25

Example 14: Find (-5)2

(-5)2 = (-5)·(-5) = 25.

**SQUARE ROOTS AND IRRATIONAL NUMBERS**.

The **square root** of a number is that number which you would take to the second power (square) to get your original number. For example, the square root of 36, written = 6, because 62 = 6·6 = 36. It is also true, of course that (-6)2 = 36 but the square root is always defined to the **principal,** or positive square root of a number, never as the negative square root.

You can always check quickly using a calculator whether a positive integer has a positive integer as a square root. For example, = 19. This is probably not a square root that you would know right away, but if you take the square root of 361 using a calculator then it will come out to exactly 19, with no decimals.

However, this is not the usual situation with the square root of a positive integer. Only some positive integers have square roots which are rational numbers. Most of them are **irrational numbers**, meaning that in decimal form they go on forever and neither terminate nor repeat. For example, if you use a calculator to find the , the calculator will say something like 1.4142135623730950488 (or possibly to a different number of decimal places depending on how wide your calculator display is.) This is **NOT**, however the exact value of . It is only a **decimal approximation,** which is an error intentionally made for the purpose of ease in writing. Since written as a decimal would go on forever it is not possible to write it exactly as a decimal. In fact, the simplest way to write exactly is precisely that, to write . So if we are asked to simplify we should answer straightforwardly that the answer is , or is ‘done.’ There is no simpler and exact way to write .

In some cases however irrational square roots can be simplified, with the goal of expressing them with the **smallest possible number** remaining under the radical. To do this, we must first make an observation, specifically that if you are taking the square root of the product of two numbers then the result can be written as the square root of the first number times the square root of the second number; i.e.

= · .

What is important though is that we specifically write it as the product of the largest **square** that we can take out times the smallest number that does not include a square. For example if we are trying to simplify we might be tempted to write 12 as 2·6, but in fact that would do us no good because neither 2 nor 6 is a square. Rather, we would write 12 as 4·3 because 4 is a square so that we can take = 2 in order to simplify. Hence,

= = · = 2 .

Example 15: Simplify .

= = = 5 .

Note that although it is also true that 50 = 5 10, this would not help us because neither 5 nor 10 is a square. In fact, if we prime factor 50 = 2·5·5 we can see right away that the square in it must be 5·5 = 25.

Example 16: Simplify -8. Note that the number -8 is already outside the radical so it will play no part in simplifying the radical but instead multiply whatever we are able to take out.

If we prime factor 98 we get 2·7·7 so the square inside of 98 must be 7·7 = 49.

-8 = -8 = -8·7 = -56 .

Example 17: Simplify 3 Prime factor 80 = 2·2·2·2·5

Note that when we prime factor we can either find 2·2 = 4 or 2·2·2·2 = 16 as a square inside of 80. However we want to choose the largest square we can to make sure we have gotten everything possible out, so we write 80 = 16·5.

3 = 3 = 3·4 = 12.

Example 18: Simplify Prime factor 70 = 2·5·7.

Note that 70 does not contain a square inside it (there is only one of every prime factor) so in fact **cannot be simplified**. We can either write it that way or just write, “**Done**.”

Example 19: Find a decimal approximation to to the thousandths place.

Recall that the thousandths place is the third place after the decimal. In this problem we are **not asked to simplify**, so the answer 10, which is the exact answer, is not what you are asked to find. Instead you take out your calculator and ask it for You get is about 22.36067977499789696409…

The third place after the decimal would be the ‘zero.’ However, when we round, we should look at the next place **after** where we are rounding. If the digit in that place is a 0-4 you round down. But if it is 5-9 you round up. Here it is a 6, so we round up. Hence = 22.361, accurate to the thousandths place.

**PROBLEM SET A-2**

In problems 1-16, write the rational numbers as decimals, either going to the last digit if the decimal terminates or using a repeating bar where needed if it repeats.

1. 2. 3. 4. 5. 6. 7. 8.

9. 10. 11. 12. 13. 14. 15. 16.

In problems 17- 32 write the decimals or fractions as percentages. Exact answers only, no rounding.l

17. .245 18. 3.11 19. .033 20. 1.625 21. 22. 23. 24.

25. .02 26. .56 27. .0004 28. .00002 29. 30. 31. 32.

In problems 33-44 write the percentages as both a decimal number and as a reduced fraction.

33. 55% 34. 68 % 35. 50% 36. 37% 37. 90% 38. 350 %

39. 14% 40. 5 % 41. 84 % 42. 400% 43. 25 % 44. 110%

In problems 45-56 write as a fraction.

45. 6 % 46. 15 % 47. 77 % 48. 33 % 49. % 50. 83 %

51. 8 % 52. 27 % 53. 5 % 54. 10 % 55. 66 % 56. %

Add or subtract the rational numbers:

57. + 58. - 59. + 60. + 61. +

62. + 63. + 64. - 65. - 66. -

67. + 68. + 69. - ( ) 70. - () 71. + -

72. + - 73. - - 74. - + 75. - + 76. 1 -

Multiply or divide the rational numbers:

77. ()() 78. ( )() 79. ()2 80. ( )2 81.( ) ()

82. ()() 83. (2)( ) 84. (12)( ) 85. ()()() 86. ()()()

87. (0)() 88. ()(0) 89. ÷ 90. ÷ 91. ÷

92. ÷ 93. ÷ 94. ÷ 95. ÷ 96. 3 ÷

97. ÷ 98. ÷ 99. ÷ 100. 0 ÷ 101. ÷ 0

Calculate the numbers:

102. 72 103. 53 104. 26 105. -42 106. (-4)2

107. (-8)2 108. -82 109. 2· 32 110. (5)3(2)2 111. ()2 (6)3

112. ()2(2)3 113. (0)(46)3

Simplify the numbers.

114. 115. 116. 3 117. -2 118.

119. 120. 121. 122. 123. 5

124. -3 125. 2 126. 127. 128. 12

Find decimal approximations accurate to the thousandths place:

129. 130. 131. 132. 133.

134.

Review problems:

In problems 135-142 give all values of the variable that satisfy the given equations.

135. |x| = 14 136. |x| = -11 137. |x| = 9 138. |x| = 32

139. |x| = 0 140. |x| = 55 141. |x| = -38 142. |x| = 13