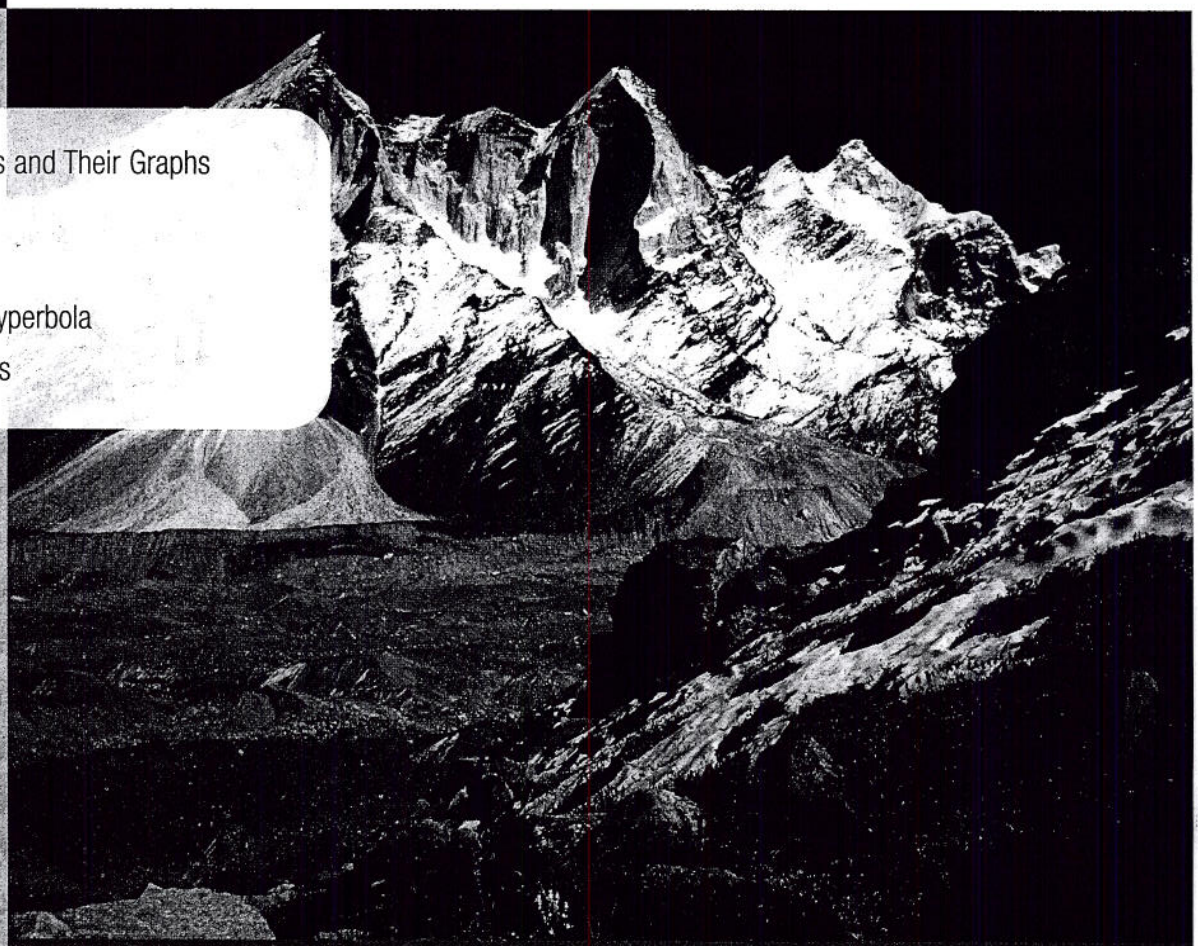
5 Rational Functions and Conic Sections



How does a business make decisions about how many units of their products to produce in order to make a profit? Suppose you were a manufacturer of graphing calculators. Your fixed or start-up cost is, let's say, \$10,000. Before you produce a single calculator, you are already spending money! Now let's assume each calculator costs you \$50 to manufacture. What is the average cost to you of producing your first hundred calculators? What formula will give you the average cost of the first x calculators?

The formula you need is a rational function of *x*. Take a look at this chapter's project at the end of the chapter. The intersecting worlds of business, finance, and economics generally all rely upon mathematics, and especially the mathematics of functions. Investigate the history of functions and the rest of mathematics at a site hosted by the British Society for the History of Mathematics, http://www.dcs.warwick.ac.uk/bshm/resources.html.

In this chapter, we are going to investigate two types of relationships. The first is derived from the polynomial functions considered in Chapter 3. Specifically, we will study a function that is the quotient of two polynomials. Although sums, differences, and products of two polynomials still produce polynomials, quotients in general tend to be more complicated. Additionally, we will examine the various curves that arise when one considers the intersection of a plane and a cone. These curves will be the graphs of an important class of relationships that are not necessarily functions.

In 1637, the French philosopher and scientist, René Descartes, developed an idea that combined the techniques of algebra with those of geometry. He created a new field of study called **analytic geometry,** an area in which one can apply the methods and equations of algebra to the solution of problems in geometry and also obtain geometric representations of algebraic equations. In Chapter 3, we observed some applications of these methods in deriving the distance and midpoint formulas. Analytic geometry is also well suited to the study of conic sections, also, since geometry plays such an important role here as well.

■ ■ ■

5.1 Rational Functions and Their Graphs

We can apply our knowledge of polynomial functions to the study of a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$. This is called a **rational function.** (You may recall that a *rational number* is the quotient of two integers.) We will assume that the polynomials $P(x)$ and $Q(x)$ have no common factors, and we will call such a rational function **irreducible.** (Show that $P(x)$ and $Q(x)$ have no common zeros.) We will also assume that $Q(x)$ is of degree 1 or higher. (If $Q(x)$ were of degree 0, it would actually be a constant; hence $f(x)$ would be a polynomial.)

Domain and Intercepts

Since $P(x)$ and $Q(x)$ are polynomials, they are both defined for all real values of x. The function f can have "problems" only where the denominator is zero. Consequently, the domain of f consists of all real numbers except those for which $Q(x) = 0$.

To find the y-intercepts of the function f, set x equal to 0 and evaluate $y = f(0)$. Should $Q(0) = 0$, then $f(0)$ is undefined and there are no y-intercepts. ($f(x)$ has at most one y-intercept.)

To find the x-intercepts, we note that $y = f(x)$ can be 0 only if the numerator $P(x)$ is zero. Therefore, the x-intercepts correspond to the roots of the polynomial equation $P(x) = 0$. Since we have assumed that f is irreducible, if r is such that $P(r) = 0$, then $Q(r) \neq 0$. In other words, $P(x)$ and $Q(x)$ have no common zeros.

EXAMPLE 1 DOMAIN AND INTERCEPTS

Find the domain and intercepts of each irreducible rational function.

a. $f(x) = \dfrac{x+1}{x-1}$ b. $g(x) = \dfrac{x^3 + 2x^2 - 3x}{x^2 - 4}$ c. $h(x) = \dfrac{x^2 - 9}{x^2 + 1}$

SOLUTION

a. The denominator is 0 when $x = 1$. Thus, the domain of f is the set of all real numbers except $x = 1$.

To find the y-intercept, we set $x = 0$ and find $y = f(0) = -1$. To find the x-intercepts, we set the numerator equal to 0 and find that $y = f(x) = 0$ when $x = -1$. Summarizing, the y-intercept is $(0, -1)$, and the x-intercept is $(-1, 0)$.

b. The denominator is 0 when $x = 2$ and when $x = -2$. The domain of g is then the set of all real numbers except $x = \pm 2$.

To find the y-intercept, we set $x = 0$ and find $y = g(0) = 0$. To find the x-intercepts, we set the numerator equal to 0 and find that

$$x^3 + 2x^2 - 3x = 0$$
$$x(x^2 + 2x - 3) = 0$$
$$x(x-1)(x+3) = 0$$

has the solutions $x = 0$, $x = 1$, and $x = -3$. Summarizing, the y-intercept is $(0, 0)$, and the x-intercepts are $(0, 0)$, $(1, 0)$, and $(-3, 0)$.

c. Since the denominator $x^2 + 1$ can never be zero (Why?), the domain of h is the set of all real numbers.

To find the y-intercept, we set $x = 0$ and find $y = h(0) = -9$. To find the x-intercepts, we set the numerator $x^2 - 9 = 0$ and find that $x = \pm 3$. Summarizing, the y-intercept is $(0, -9)$, and the x-intercepts are $(3, 0)$ and $(-3, 0)$.

✔ Progress Check

Find the domain and intercepts of each rational function.

a. $S(x) = \dfrac{x-3}{2x^2 - 3x - 2}$ b. $T(x) = \dfrac{5}{x^4 + x^2 + 5}$

Answers

a. *Domain:* all real numbers except $x = -\frac{1}{2}$, $x = 2$; y-intercept: $(0, \frac{3}{2})$; x-intercept: $(3, 0)$.

b. *Domain:* all real numbers; y-intercept: $(0, 1)$; no x-intercept.

Graphing $\dfrac{k}{x}$ and $\dfrac{k}{x^2}$

We begin the study of graphs of rational functions by considering examples in which the numerator is a constant.

x	$y = \frac{1}{x}$
$\frac{1}{1000}$	1000
$\frac{1}{100}$	100
$\frac{1}{10}$	10
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

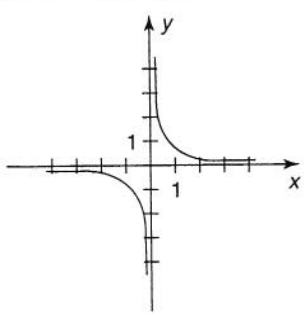


FIGURE 1 Graph of $y = \dfrac{1}{x}$

EXAMPLE 2 RATIONAL FUNCTIONS WITH CONSTANT NUMERATORS

Sketch the graph of the function

$$f(x) = \frac{1}{x}$$

SOLUTION

Domain: The denominator is 0 when $x = 0$. Thus, the domain of f is the set of all real numbers except $x = 0$.

Intercepts: There are no x-intercepts or y-intercepts.

Symmetry: The graph of f is symmetric with respect to the origin since the equation remains unchanged when x and y are replaced by $-x$ and $-y$, respectively. Therefore, we only need to plot those points corresponding to positive values of x. We sketch the graph of $y = \frac{1}{x}$ as shown in Figure 1.

EXAMPLE 3 RATIONAL FUNCTIONS WITH CONSTANT NUMERATORS

Sketch the graph of the function

$$f(x) = \frac{1}{x^2}$$

x	$y = \frac{1}{x^2}$
$\frac{1}{1000}$	1,000,000
$\frac{1}{100}$	10,000
$\frac{1}{10}$	100
1	1
2	$\frac{1}{4}$
4	$\frac{1}{16}$

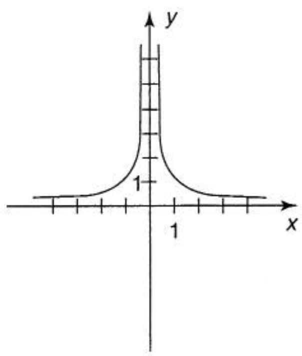


FIGURE 2 Graph of $y = \dfrac{1}{x^2}$

SOLUTION

Domain: The denominator is 0 when $x = 0$. Thus, the domain of f is the set of all real numbers except $x = 0$.

Intercepts: There are no x-intercepts or y-intercepts.

Symmetry: The graph of f is symmetric with respect to the y-axis since the equation remains unchanged when x is replaced by $-x$. Therefore, we only need consider positive values of x. We sketch the graph of $y = \frac{1}{x^2}$ as shown in Figure 2.

Asymptotes

The graphs in Figure 1 and Figure 2 illustrate an important concept: the graphs appear to approach specific horizontal and vertical lines without ever touching them. Such lines play an important role in the graphs of many functions, and we may define them in the following intuitive way.

A line is said to be an **asymptote** of a graph if the graph gets closer and closer to the line as we move farther and farther out along the line.

Note the behavior of the graphs in Figures 1 and 2 as x gets closer and closer to 0. Both graphs approach the y-axis, and we say that the line $x = 0$ (the y-axis) is a **vertical asymptote** for each graph. Similarly, as $|x|$ gets extremely large, both graphs approach the x-axis, and we say that the line $y = 0$ (the x-axis) is a **horizontal asymptote** for each graph.

EXAMPLE 4 USING ASYMPTOTES IN GRAPHING

Sketch the graph of the rational function

$$F(x) = \frac{1}{(x-1)}$$

SOLUTION

If we compare $F(x) = \dfrac{1}{x-1}$ with the function from Example 2, $f(x) = \dfrac{1}{x}$, we see that

$$F(x) = f(x-1)$$

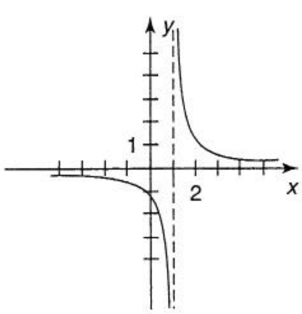


FIGURE 3 Graph of $F(x) = \dfrac{1}{x-1}$

From Section 3.3, we may observe that the graph of $F(x)$ is that of $f(x)$ shifted 1 unit to the right, as shown in Figure 3. Note that the vertical asymptote has also been shifted or *translated* 1 unit to the right. (Shifting the graph to the right or left leaves the horizontal asymptote unchanged.)

EXAMPLE 5 USING ASYMPTOTES IN GRAPHING

Sketch the graph of the rational function

$$F(x) = \frac{3}{(x+2)^2}$$

SOLUTION

If we compare

$$F(x) = \frac{3}{(x+2)^2}$$

with the function from Example 3, $f(x) = \frac{1}{x^2}$, we see that

$$F(x) = 3f(x+2)$$

From Section 3.3, we may observe that the graph of $F(x)$ is that of $f(x)$ shifted 2 units to the left and stretched by a factor of 3, as shown in Figure 4. Note that the vertical asymptote has also been translated 2 units to the left.

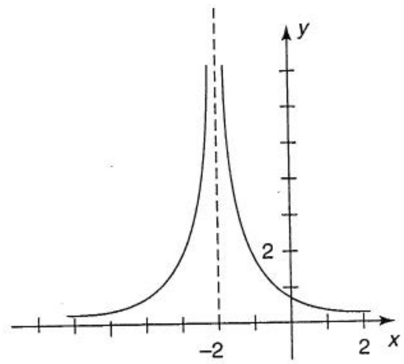


FIGURE 4 Graph of $F(x) = \frac{3}{(x+2)^2}$

Since the asymptotes play an important role in graphing rational functions, it is useful to find a procedure for locating them. The graphs in each of the preceding figures of this chapter indicate that the functions *increase without bound,* **approaching** ∞, or *decrease without bound,* **approaching** $-\infty$, as the curve approaches a vertical asymptote. Note that in these cases, the absolute value of the denominator of the quotient gets closer and closer to 0. The following theorem provides the means for finding all vertical asymptotes.

Vertical Asymptote Theorem

The graph of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

has a vertical asymptote at $x = r$ if r is a real root of $Q(x)$ but not of $P(x)$.

EXAMPLE 6 VERTICAL ASYMPTOTES

Find the vertical asymptotes of the function

$$f(x) = \frac{2}{x^3 - 2x^2 - 3x}$$

SOLUTION

Factoring the denominator, we have

$$f(x) = \frac{2}{x(x+1)(x-3)}$$

Therefore, $x = 0$, $x = -1$, and $x = 3$ are the vertical asymptotes of $f(x)$.

To find the horizontal asymptotes of a function, we must examine the behavior of that function as x approaches ∞ and as x approaches $-\infty$, that is, as $|x|$ *increases without bound.* Recall the expression

$$\frac{k}{x^n}$$

where k is a constant and n is a positive integer. This expression becomes very small as $|x|$ becomes very large. In other words, $\frac{k}{x^n}$ approaches 0 as $|x|$ approaches ∞.

EXAMPLE 7 HORIZONTAL ASYMPTOTES

Find the horizontal asymptotes of the function

$$f(x) = \frac{2}{x^3 - 2x^2 - 3x}$$

SOLUTION

If we factor out x^3 from the denominator, we have

$$f(x) = \frac{2}{x^3\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)} = \left(\frac{2}{x^3}\right)\left(\frac{1}{1 - \frac{2}{x} - \frac{3}{x^2}}\right)$$

As $|x|$ approaches ∞, the terms $\frac{2}{x^3}$, $-\frac{2}{x}$, and $-\frac{3}{x^2}$ approach 0. Therefore, $f(x)$ approaches 0 as $|x|$ approaches ∞; and hence, $y = 0$ is the only horizontal asymptote.

EXAMPLE 8 USING ASYMPTOTES IN GRAPHING

Sketch the graph of the function

$$f(x) = \frac{2}{x^3 - 2x^2 - 3x}$$

SOLUTION

Since we can write

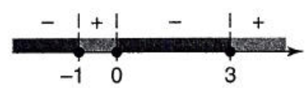
$$f(x) = \frac{2}{x(x+1)(x-3)}$$

we determine that the critical values of $f(x)$ are -1, 0, and 3. (See the Critical Value Method in Section 2.5)

Solution by the Critical Value Method

Interval	Test Point	Substitution	Sign
$x < -1$	$x = -2$	$f(-2) < 0$	$-$
$-1 < x < 0$	$x = -\frac{1}{2}$	$f(-\frac{1}{2}) > 0$	$+$
$0 < x < 3$	$x = 1$	$f(1) < 0$	$-$
$x > 3$	$x = 4$	$f(4) > 0$	$+$

These results are summarized as follows:



If $x < -1$ and x approaches -1, then $f(x)$ approaches $-\infty$. If $-1 < x < 0$ and x approaches -1, then $f(x)$ approaches ∞. Using a similar analysis at each point corresponding to a vertical asymptote, we may draw the following partial graphs of $f(x)$ as shown in Figure 5. From Example 7, we have that $y = 0$ is the horizontal asymptote. Combining this observation with the portions of the graph of $f(x)$ sketched in Figure 5 leads to the graph of $f(x)$ sketched in Figure 6.

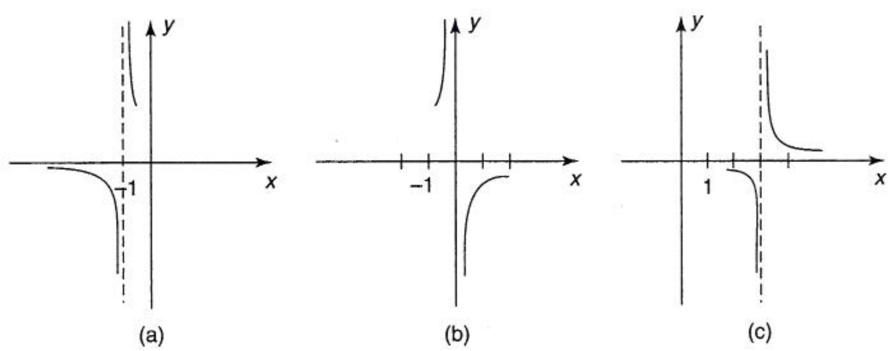


FIGURE 5 Partial Graphs of $f(x) = \frac{2}{x(x+1)(x-3)}$

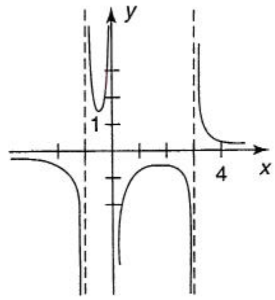


FIGURE 6 Graph of $f(x) = \dfrac{2}{x(x+1)(x-3)}$

EXAMPLE 9 HORIZONTAL ASYMPTOTES

Find the horizontal asymptotes of the function

$$f(x) = \frac{2x^2 - 5}{3x^2 + 2x - 4}$$

SOLUTION

We illustrate the steps of the procedure as follows:

Horizontal Asymptotes

Step 1. Factor out the highest power of x found in the numerator, and factor out the highest power of x found in the denominator.

Step 1.

$$f(x) = \frac{x^2\left(2 - \frac{5}{x^2}\right)}{x^2\left(3 + \frac{2}{x} - \frac{4}{x^2}\right)}$$

Step 2. Since we are interested in large values of $|x|$, we may cancel common factors in the numerator and denominator.

Step 2.

$$f(x) = \frac{2 - \frac{5}{x^2}}{3 + \frac{2}{x} - \frac{4}{x^2}}, \quad x \neq 0$$

Step 3. As $|x|$ increases, terms of the form $\frac{k}{x^n}$ approach 0 and may be ignored.

Step 3. The terms $-\frac{5}{x^2}$, $\frac{2}{x}$ and $-\frac{4}{x^2}$ approach 0 as $|x|$ approaches ∞.

Step 4. If $f(x)$ approaches some number c, then $y = c$ is a horizontal asymptote. Otherwise, there is no horizontal asymptote.

Step 4. Ignoring these terms, we have $y = \frac{2}{3}$ as the horizontal asymptote.

EXAMPLE 10 HORIZONTAL ASYMPTOTES

Find the horizontal asymptotes of the function

$$f(x) = \frac{2x^3 + 3x - 2}{x^2 + 5}$$

SOLUTION

Factoring, we have

$$f(x) = \frac{x^3\left(2 + \frac{3}{x^2} - \frac{2}{x^3}\right)}{x^2\left(1 + \frac{5}{x^2}\right)}$$

$$= \frac{x\left(2 + \frac{3}{x^2} - \frac{2}{x^3}\right)}{1 + \frac{5}{x^2}}, \quad x \neq 0$$

As $|x|$ increases, the terms $\frac{3}{x^2}$, $-\frac{2}{x^3}$, and $\frac{5}{x^2}$ approach zero and can be ignored. Therefore, as $|x|$ increases, $f(x)$ approaches $2x$. However, as $|x|$ approaches ∞, $f(x)$ does not approach some number c. In fact, $|y|$ approaches ∞ as $|x|$ approaches ∞. Thus, there is no horizontal asymptote.

The following theorem can be proved by utilizing the procedure of Example 9.

Horizontal Asymptote Theorem

The graph of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

has a horizontal asymptote if the degree of $P(x)$ is less than or equal to the degree of $Q(x)$.

Note that the graph of a rational function may have many vertical asymptotes but at most one horizontal asymptote. A more specific version of the Horizontal Asymptote Theorem can be found in Exercises 28 and 29.

✔ Progress Check

Determine the horizontal asymptote of the graph of each function.

a. $f(x) = \dfrac{x - 1}{2x^2 + 1}$

b. $g(x) = \dfrac{4x^2 - 3x + 1}{-3x^2 + 1}$

c. $h(x) = \dfrac{3x^3 - x + 1}{2x^2 - 1}$

Answers

a. $y = 0$ b. $y = -\dfrac{4}{3}$ c. no horizontal asymptote

Sketching Graphs

We now summarize the information that can be gathered in preparation for sketching the graph of a rational function:

- symmetry with respect to the axes and the origin
- x-intercepts, y-intercepts
- vertical asymptotes
- horizontal asymptotes
- brief table of values including points near the vertical asymptotes

EXAMPLE 11 GRAPHING RATIONAL FUNCTIONS

Sketch the graph of

$$f(x) = \frac{x^2}{x^2 - 1}$$

SOLUTION

Symmetry: Replacing x with $-x$ results in the same equation, establishing symmetry with respect to the y-axis.

Intercepts: Setting $x = 0$, we obtain the y-intercept $y = 0$. Setting $y = f(x) = 0$ yields the x-intercept $x = 0$. Therefore, the point $(0, 0)$ is both the x- and y-intercept.

Vertical asymptotes: Setting the denominator equal to zero, we find that $x = 1$ and $x = -1$ are vertical asymptotes of the graph of f.

Horizontal asymptotes: We note that

$$f(x) = \frac{x^2}{x^2\left(1 - \frac{1}{x^2}\right)} = \frac{1}{1 - \frac{1}{x^2}}, \quad x \neq 0$$

As $|x|$ gets larger and larger, $\frac{1}{x^2}$ approaches 0 and the values of $f(x)$ approach 1. Thus, $y = 1$ is the horizontal asymptote.

We determine the critical values of $f(x)$ to be 0 and ± 1. Our analysis of the behavior of $f(x)$ in the various intervals yields

Following the technique used in Example 8 and plotting a few points, we sketch the graph in Figure 7.

x	y
$\frac{1}{2}$	-0.33
$\frac{3}{4}$	-1.29
$\frac{5}{4}$	2.78
$\frac{3}{2}$	1.80
2	1.33

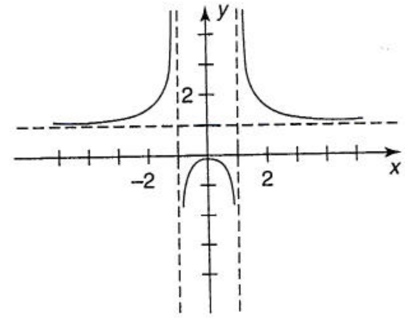


FIGURE 7 Graph of $f(x) = \dfrac{x^2}{x^2 - 1}$

✔ Progress Check

Find the horizontal and vertical asymptotes, and the x- and y-intercepts. Sketch the graph of the function

$$f(x) = \frac{x^2 - x - 6}{x^2 - 2x}$$

Answers

The horizontal asymptote is $y = 1$. The vertical asymptotes are $x = 0$ and $x = 2$. There is no y-intercept, and the x-intercepts are $(3, 0)$ and $(-2, 0)$. We sketch the graph of $f(x)$ in Figure 8.

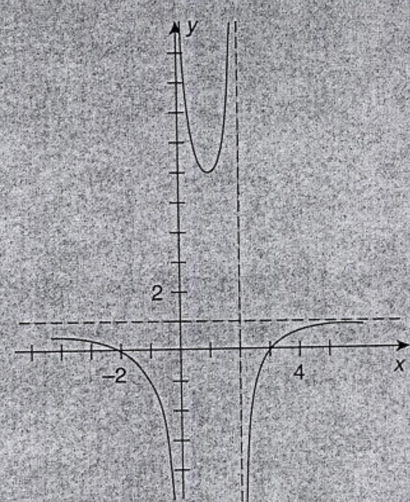


FIGURE 8 Graph of $f(x) = \dfrac{x^2 - x - 6}{x^2 - 2x}$

Reducible Rational Functions

We conclude this section with an example of a **reducible** rational function, that is, one in which the numerator and denominator have a factor in common other than a constant. Reducible rational functions are often used to illustrate functions that have "holes" in their graphs. Such functions are not continuous at these holes.

EXAMPLE 12 GRAPHING REDUCIBLE RATIONAL FUNCTIONS

Sketch the graph of the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

SOLUTION

We observe that

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1, \qquad x \neq 1$$

Thus, the graph of the function $f(x)$ coincides with the line $y = x + 1$, with the exception that $f(x)$ is undefined at $x = 1$. We sketch the graph of $f(x)$ in Figure 9.

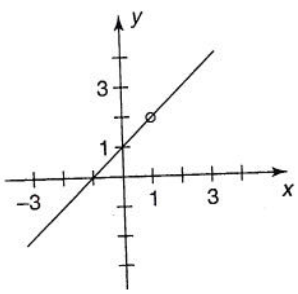


FIGURE 9 Graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$

✔ Progress Check

Sketch the graph of the function

$$f(x) = \frac{4 - x^2}{x + 2}$$

Answer

Figure 10 is the graph of $f(x)$.

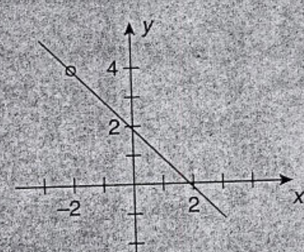


FIGURE 10 Graph of $f(x) = \dfrac{4 - x^2}{x + 2}$

In discussing graphs and their asymptotes, we did not consider the possibility of an *oblique,* or *slanted,* asymptote. This is an asymptote that is neither horizontal nor vertical. A rational function has an oblique asymptote if the degree of the numerator is one more than the degree of the denominator.

EXAMPLE 13 FINDING OBLIQUE ASYMPTOTES

Find the oblique asymptote of

$$f(x) = \frac{x^2 - x}{x + 1}$$

SOLUTION

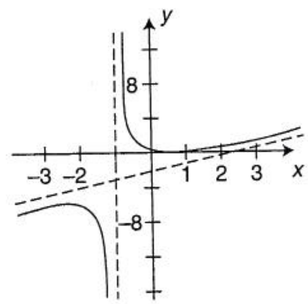
Since the degree of the numerator − degree of the denominator = 2 − 1 = 1 there is an oblique asymptote. After we perform the division, we obtain

$$f(x) = \frac{x^2 - x}{x + 1} = x - 2 + \frac{2}{x + 1}$$

As $|x|$ gets larger and larger, $\frac{2}{x+1}$ approaches 0 and $f(x)$ gets closer and closer to its oblique asymptote, the line

$$y = x - 2$$

as shown in the following figure:

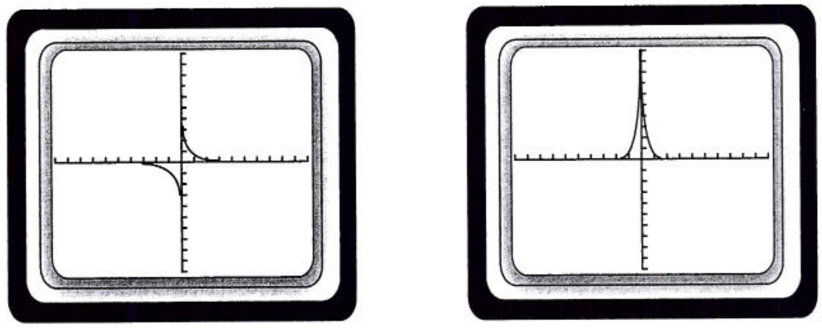


Graphing Calculator Power User's Corner

Graphing Rational Functions

Rational functions present several challenges to users of graphing calculators. It is important to be knowledgeable about these functions to properly interpret the display on your graphing calculator.

For example, graphs of rational functions sometimes seem to be "swallowed up" by the x-axis, as in the graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ shown in the default viewing rectangle in Figure 11. This lack of detail is due to the plotting mechanism of the calculator. When $|y|$ becomes very close to zero, the calculator plots the function *on* the x-axis instead of *near* the x-axis. Hence, these points become "invisible" to the user. One way to see more detail is to shift the graphs up 1 unit, that is, graph $y = \frac{1}{x} + 1$ and $y = \frac{1}{x^2} + 1$. (Graph the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ in the EQUAL viewing rectangle and note any difference.)



(a) $f(x) = \frac{1}{x}$, $-10 \le X \le 10$, $-10 \le Y \le 10$, XSCL = 1, YSCL = 1

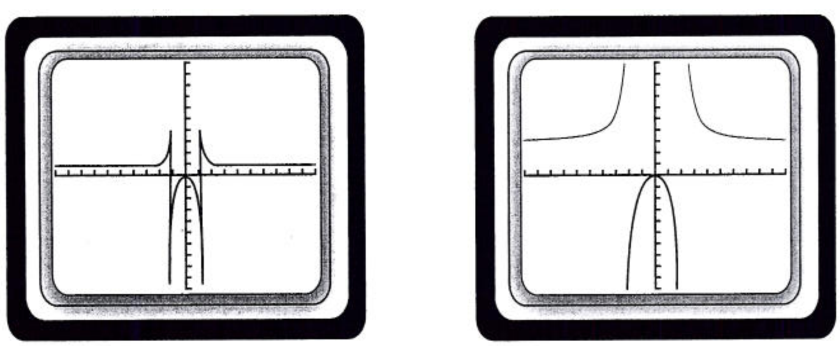
(b) $g(x) = \frac{1}{x^2}$, $-10 \le X \le 10$, $-10 \le Y \le 10$, XSCL = 1, YSCL = 1

FIGURE 11 Graphs of Rational Functions

Vertical asymptotes pose the greatest difficulty for the graphing calculator. Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}$$

from Example 11 shown in the two viewing rectangles of Figure 12. The asymptotes seem to be included in the graph in the default viewing rectangle. This happens because the calculator has not tried to evaluate $f(x)$ at $x = \pm 1$ and does not "know" that the function is undefined for these values. In the process of connecting the points that have been plotted, the calculator seems to draw the asymptotes. Graphing in a "point plotting" mode eliminates the asymptotes in this viewing rectangle, but it does not necessarily result in a better representation.



(a) TENS Viewing Rectangle, XSCL = 1, YSCL = 1

(b) EQUAL Viewing Rectangle, XSCL = 1, YSCL = 1

FIGURE 12 Graph of $f(x) = \dfrac{x^2}{x^2 - 1}$

For this example, the EQUAL viewing rectangle presents a good representation of the function without drawing the asymptotes. It is the responsibility of the user to determine that this function has vertical asymptotes at $x = \pm 1$. However, the function

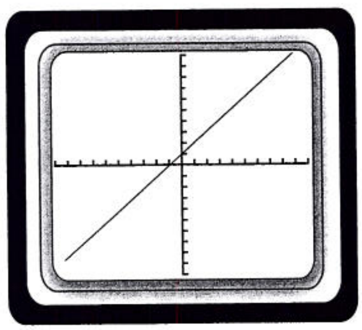
$$g(x) = \frac{x^2}{x^2 - 7}$$

has vertical asymptotes at $x = \pm\sqrt{7}$. Yet, there is no viewing rectangle that displays this function without graphing the asymptotes.

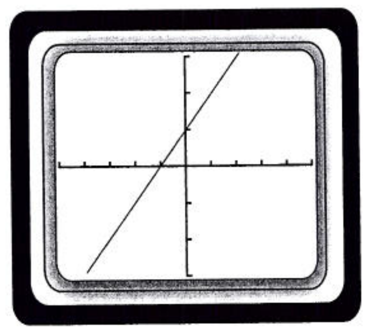
Finally, consider rational functions with "holes" in their graphs, such as Example 12, where

$$f(x) = \frac{x^2 - 1}{x - 1}$$

As shown in Figure 13(a), graphing this function in the default viewing rectangle does not indicate the "hole" at the point (1, 2) because the calculator does not attempt to plot a point for $x = 1$. However, in Figure 13(b), we see that using the EQUAL viewing rectangle does produce a visible "hole" in the graph.



(a) TENS Viewing Rectangle, XSCL = 1, YSCL = 1



(b) EQUAL Viewing Rectangle, XSCL = 1, YSCL = 1

FIGURE 13 Graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$

Although the graphing calculator is helpful in determining the graph of a rational function, a careful analysis of the behavior of the function is necessary to obtain a correct and complete representation.

Exercise Set 5.1

In Exercises 1–6, determine the domain and intercepts of the given function.

1. $f(x) = \dfrac{x^2}{x - 1}$
2. $f(x) = \dfrac{x - 1}{x^2 + x - 2}$
3. $g(x) = \dfrac{x^2 + 1}{x^2 - 2x}$
4. $g(x) = \dfrac{x^2 + 2}{x^2 - 2}$
5. $F(x) = \dfrac{x^2 - 3}{x^2 + 3}$
6. $T(x) = \dfrac{3x + 2}{2x^3 - x^2 - x}$

In Exercises 7–21, determine the vertical and horizontal asymptotes of the graph of the given function. Sketch the graph. Then, determine appropriate WINDOW values, and check your answer using your graphing calculator. Choose the viewing rectangle carefully to obtain a correct and