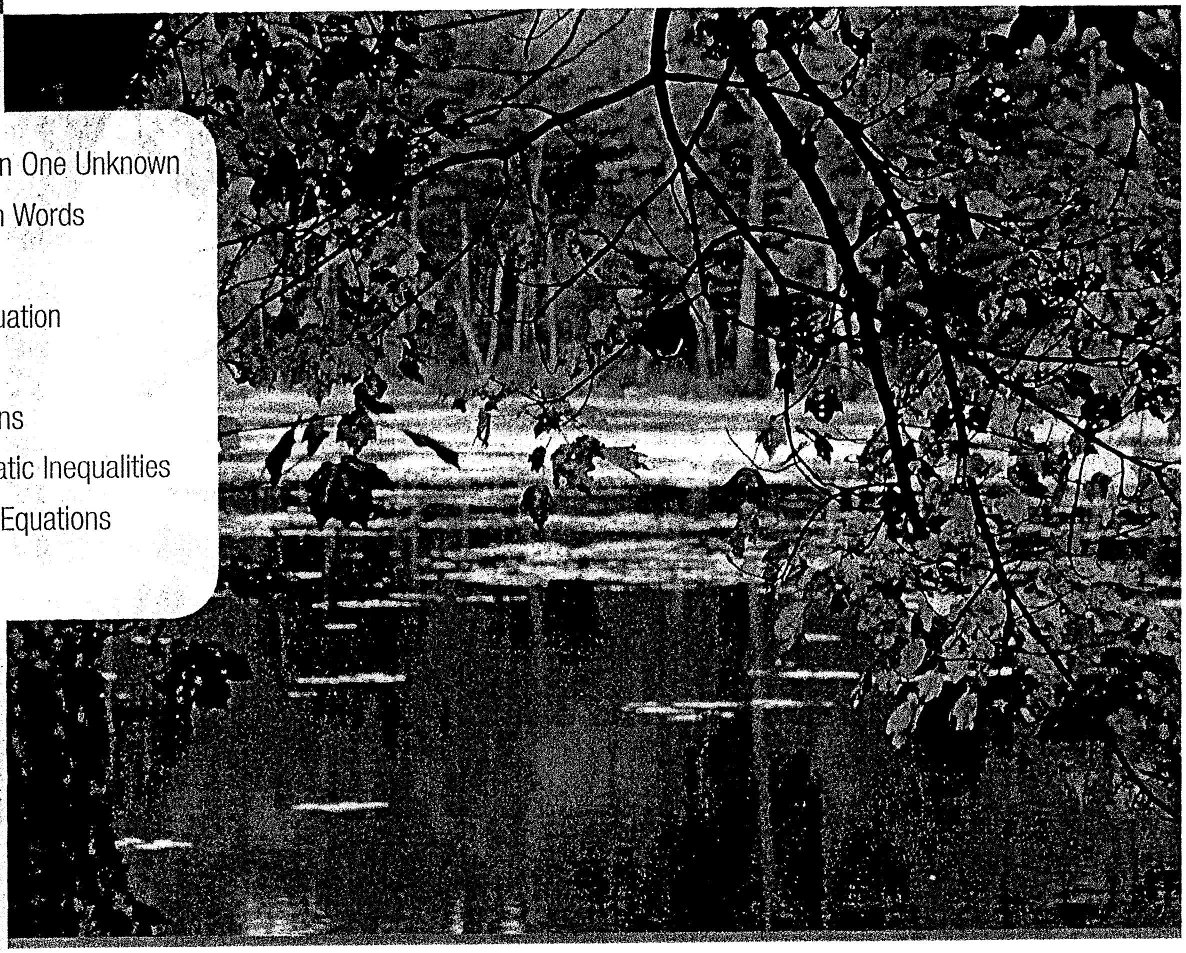
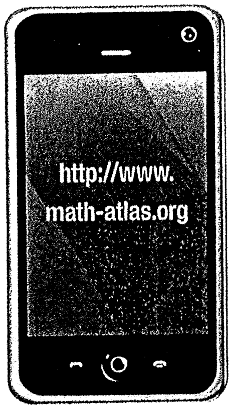
2 Equations and Inequalities



The Internet is a short form of the word "internetworking." The Internet is a vast data network, with humble origins in the 1960s. A network, whether it connects computers or people, is just a way of facilitating communication. In a **full-mesh network**, elements are linked pairwise—that is, any two elements in the system are linked directly, without intermediary.

How many elements (users) could be linked in a full-mesh network with 190 two-way links? The answer to this problem is found by solving a quadratic equation (see the Chapter Project). This chapter will show you how.



Explore the Internet for its many mathematical offerings! Check out a site which is organized according to the Mathematics Subject Classification created by the American Mathematical Society. Look up graph theory to learn more about networking.

A major concern of algebra is the solution of equations. Does a given equation have a solution? Is it possible for an equation to have more than one solution? Is there a procedure for solving an equation? In this chapter we will explore the answers to these questions for polynomial equations of the first and second degree. We will also see that the ability to solve equations enables us to tackle a wide variety of applications and word problems.

Linear inequalities also play an important role in solving word problems. For example, if we are required to combine food products in such a way that a specified minimum daily requirement for various nutrients is provided, we need to use inequalities. Many important industries, including steel and petroleum, use computers daily to solve problems that involve thousands of inequalities. The solutions to such problems enable a company to optimize its "product mix" and its profitability.

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2.1 Linear Equations in One Unknown

Solving Equations

Expressions of the form

$$x - 2 = 0 \qquad x^2 - 9 = 0 \qquad 3(2x - 5) = 3$$

$$2x + 5 = \sqrt{x - 7} \qquad \frac{1}{2x + 3} = 5 \qquad x^3 - 3x^2 = 32$$

are examples of equations in the unknown x. An **equation** states that two algebraic expressions are equal. We refer to these expressions as the **left-hand side** and the **right-hand side** of the equation.

Our task is to find values of the unknown for which the equation is satisfied. These values are called **solutions** or **roots** of the equation, and the set of all solutions is called the **solution set**. For example, 2 is a solution of the equation $3x - 1 = 5$ since $3(2) - 1 = 5$. However, -2 is *not* a solution since $3(-2) - 1 \neq 5$.

Equations that do not have solutions in one number system may have solutions in a larger number system. For example, the equation $2x - 5 = 0$ has no integer solutions but does have a solution among the rational numbers, namely $\frac{5}{2}$. Similarly, the equation $x^2 = -4$ has no solutions among the real numbers but does have solutions if we consider complex numbers, namely $2i$ and $-2i$. The solution sets of these two equations are $\{\frac{5}{2}\}$ and $\{2i, -2i\}$, respectively.

Identities and Conditional Equations

We say that an equation is an **identity** if it is true for every real number for which both sides of the equation are defined. For example, the equation

$$x^2 - 1 = (x + 1)(x - 1)$$

is an identity because it is true for all real numbers. (Try any number and check that this equation holds.) The equation

$$x - 5 = 3$$

is only true when $x = 8$. (Try any number not equal to 8 and check that this equation does not hold.) An equation such as $x - 5 = 3$, which is not true for all values of x, is called a **conditional equation.**

When we say that we want to "solve an equation," we mean that we want to find *all* solutions or roots. If we can replace an equation with another, simpler equation that has the same solutions, we will have an approach to solving equations. Equations having the same solutions are called **equivalent equations.** For example, $3x - 1 = 5$ and $3x = 6$ are equivalent equations because it can be shown that $\{2\}$ is the solution set of both equations.

There are two important rules that allow us to replace an equation with an equivalent equation.

Equivalent Equations

The solutions of a given equation are not affected by the following operations:

1. addition (or subtraction) of the same number or expression on both sides of the equation
2. multiplication (or division) by the same number, different from 0, on both sides of the equation

EXAMPLE 1 SOLVING EQUATIONS

Solve $3x + 4 = 13$.

SOLUTION

We apply the preceding rules to this equation. The strategy is to isolate x, so we *subtract 4 from both sides of the equation.*

$$3x + 4 - 4 = 13 - 4$$
$$3x = 9$$

Dividing both sides by 3, we obtain the solution

$$x = 3$$

We check by substitution to make sure that 3 does, indeed, satisfy the original equation.

$$\begin{aligned} \text{left-hand side} &= 3x + 4 \qquad \text{right-hand side} = 13 \\ &= 3(3) + 4 \\ &= 13 \end{aligned}$$

Although $x = 3$ is an equation that is *equivalent* to the original equation, in common usage we say that $3x + 4 = 13$ "has the solution $x = 3$."

When the given equation contains rational expressions, we eliminate fractions by first multiplying by the least common denominator of all fractions present. This technique is illustrated in Examples 2, 3, and 4.

EXAMPLE 2 **SOLVING EQUATIONS**

Solve the equation.

$$\frac{5}{6}x - \frac{4}{3} = \frac{3}{5}x + 1$$

SOLUTION

We first eliminate fractions by multiplying both sides of the equation by the LCD of all fractions, which is 30.

$$\left(\frac{5}{6}x - \frac{4}{3}\right)(30) = \left(\frac{3}{5}x + 1\right)(30)$$

$$25x - 40 = 18x + 30$$

$$7x = 70$$

$$x = 10$$

Verify that $x = 10$ is a solution of the original equation.

✓ Progress Check

Solve and check.

a. $-\frac{2}{3}(x - 5) = \frac{3}{2}(x + 1)$ b. $\frac{1}{3}x + 2 - 3\left(\frac{x}{2} + 4\right) = 2\left(\frac{x}{4} - 1\right)$

Answers

a. $\frac{11}{13}$ b. $-\frac{24}{5}$

Solving Linear Equations

The equations we have solved are all of the first degree and involve only one unknown. Such equations are called **first-degree equations in one unknown,** or more simply, **linear equations.** The general form of such equations is

$$ax + b = 0$$

where a and b are any real numbers and $a \neq 0$. Let us see how to solve this equation.

$$\begin{aligned} ax + b &= 0 \\ ax + b - b &= 0 - b && \text{Subtract } b \text{ from both sides.} \\ ax &= -b \\ \frac{ax}{a} &= \frac{-b}{a} && \text{Divide both sides by } a \neq 0. \\ x &= -\frac{b}{a} \end{aligned}$$

We verify that this is a solution.

$$a\left(-\frac{b}{a}\right) + b = 0$$

Furthermore, it can be shown that this is the only solution. We have thus obtained the following result:

Roots of a Linear Equation

The linear equation $ax + b = 0$, $a \neq 0$, has exactly one solution:

$$x = -\frac{b}{a}$$

Sometimes we are led to linear equations in the course of solving other equations. The following example illustrates this situation.

EXAMPLE 3 SOLVING EQUATIONS

Solve.

$$\frac{5x}{x + 3} - 3 = \frac{1}{x + 3}$$

SOLUTION

The LCD of all fractions is $x + 3$. Multiplying both sides of the equation by $x + 3$ to eliminate fractions, we obtain

$$\begin{aligned} 5x - 3(x + 3) &= 1 \\ 5x - 3x - 9 &= 1 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

Checking the solution, we have

$$\text{left-hand side} = \frac{5x}{x+3} - 3 = \frac{5(5)}{5+3} - 3 = \frac{25}{8} - 3 = \frac{25}{8} - \frac{24}{8} = \frac{1}{8}$$

$$\text{right-hand side} = \frac{1}{x+3} = \frac{1}{5+3} = \frac{1}{8}$$

We said earlier that multiplication (or division) of both sides of an equation by any nonzero number results in an equivalent equation. What happens if we multiply or divide an equation by an expression that contains an unknown? In Example 3, this procedure worked and gave us a solution. However, this may not always be so since the answer we obtain may produce a zero denominator when substituted back into the original equation. Therefore, the following rule must be carefully observed:

Multiplying by an Unknown

Multiplication (or division) by the same expression on both sides of an equation may result in an equation that is *not* equivalent to the original equation. Always verify that the answer obtained to the subsequent equation is, indeed, a solution to the original equation.

EXAMPLE 4 EQUATIONS WITH NO SOLUTION

Solve and check.

$$\frac{8x+1}{x-2} + 4 = \frac{7x+3}{x-2}$$

SOLUTION

The LCD of all fractions is $x - 2$. Multiplying both sides of the equation by $x - 2$, we eliminate fractions and obtain

$$8x + 1 + 4(x-2) = 7x + 3$$
$$8x + 1 + 4x - 8 = 7x + 3$$
$$5x = 10$$
$$x = 2$$

Checking our answer, we find that $x = 2$ is not a solution since substituting $x = 2$ in the original equation yields a denominator of zero. We conclude that the given equation has no solution.

✔ Progress Check

Solve and check.

a. $\frac{3}{x} - 1 = \frac{1}{2} - \frac{6}{x}$ b. $-\frac{2x}{x+1} = 1 + \frac{2}{x+1}$

Answers

a. $x = 6$ b. no solution

EXAMPLE 5 EQUATIONS WITH NO SOLUTION

Solve the equation $2x + 1 = 2x - 3$.

SOLUTION

Subtracting $2x$ from both sides, we have

$$2x + 1 - 2x = 2x - 3 - 2x$$
$$1 = -3$$

This equivalent equation is a contradiction, so we conclude that the given equation has no solution.

Exercise Set 2.1

In Exercises 1–4, determine whether the given statement is true (T) or false (F).

1. $x = -5$ is a solution of $2x + 3 = -7$.
2. $x = \frac{5}{2}$ is a solution of $3x - 4 = \frac{5}{2}$.
3. $x = \frac{6}{4-k}$, $k \neq 4$ is a solution of $kx + 6 = 4x$.
4. $x = \frac{7}{3k}$, $k \neq 0$ is a solution of $2kx + 7 = 5x$.

In Exercises 5–24, solve the given linear equation and check your answer.

5. $3x + 5 = -1$
6. $5r + 10 = 0$
7. $2 = 3x + 4$
8. $\frac{1}{2}s + 2 = 4$
9. $\frac{3}{2}t - 2 = 7$
10. $-1 = -\frac{2}{3}x + 1$
11. $0 = -\frac{1}{2}a - \frac{2}{3}$
12. $4r + 4 = 3r - 2$
13. $-5x + 8 = 3x - 4$
14. $2x - 1 = 3x + 2$
15. $-2x + 6 = -5x - 4$
16. $6x + 4 = -3x - 5$
17. $2(3b + 1) = 3b - 4$

18. $-3(2x + 1) = -8x + 1$

19. $4(x - 1) = 2(x + 3)$

20. $-3(x - 2) = 2(x + 4)$

21. $2(x + 4) - 1 = 0$

22. $3a + 2 - 2(a - 1) = 3(2a + 3)$

23. $-4(2x + 1) - (x - 2) = -11$

24. $3(a + 2) - 2(a - 3) = 0$

Solve for x in Exercises 25–28.

25. $kx + 8 = 5x$

26. $8 - 2kx = -3x$

27. $2 - k + 5(x - 1) = 3$

28. $3(2 + 3k) + 4(x - 2) = 5$

Solve and check in Exercises 29–44.

29. $\dfrac{x}{2} = \dfrac{5}{3}$

30. $\dfrac{3x}{4} - 5 = \dfrac{1}{4}$

31. $\dfrac{2}{x} + 1 = \dfrac{3}{x}$

32. $\dfrac{5}{a} - \dfrac{3}{2} = \dfrac{1}{4}$

33. $\dfrac{2y - 3}{y + 3} = \dfrac{5}{7}$

34. $\dfrac{1 - 4x}{1 - 2x} = \dfrac{9}{8}$

35. $\dfrac{1}{x - 2} + \dfrac{1}{2} = \dfrac{2}{x - 2}$

36. $\dfrac{4}{x - 4} - 2 = \dfrac{1}{x - 4}$

37. $\dfrac{2}{x - 2} + \dfrac{2}{x^2 - 4} = \dfrac{3}{x + 2}$

38. $\dfrac{3}{x - 1} + \dfrac{2}{x + 1} = \dfrac{5}{x^2 - 1}$

39. $\dfrac{x}{x - 1} - 1 = \dfrac{3}{x + 1}$

40. $\dfrac{2}{x - 2} + 1 = \dfrac{x + 2}{x - 2}$

41. $\dfrac{4}{b} - \dfrac{1}{b + 3} = \dfrac{3b + 2}{b^2 + 2b - 3}$

42. $\dfrac{3}{x^2 - 2x} + \dfrac{2x - 1}{x^2 + 2x - 8} = \dfrac{2}{x + 4}$

43. $\dfrac{3r + 1}{r + 3} + 2 = \dfrac{5r - 2}{r + 3}$

44. $\dfrac{2x - 1}{x - 5} + 3 = \dfrac{3x - 2}{5 - x}$

In Exercises 45–48, indicate whether the equation is an identity (I) or a conditional equation (C).

45. $x^2 + x - 2 = (x + 2)(x - 1)$

46. $(x - 2)^2 = x^2 - 4x + 4$

47. $2x + 1 = 3x - 1$

48. $3x - 5 = 4x - x - 2 - 3$

In Exercises 49–54, write (T) if the equations within each exercise are all equivalent equations and (F) if they are not equivalent.

49. $2x - 3 = 5$ $\quad 2x = 8$ $\quad x = 4$

50. $5(x - 1) = 10$ $\quad x - 1 = 2$ $\quad x = 3$

51. $x(x - 1) = 5x$ $\quad x - 1 = 5$ $\quad x = 6$

52. $x = 5$ $\quad x^2 = 25$

53. $3(x^2 + 2x + 1) = -6$ $\quad x^2 + 2x + 1 = -2$
$(x + 1)^2 = -2$

54. $(x + 3)(x - 1) = x^2 - 2x + 1$
$(x + 3)(x - 1) = (x - 1)^2$ $\quad x + 3 = x - 1$

55. Write repeating decimal fractions as rational equivalents.

Example:

$$\begin{aligned} N &= 0.1515\ldots = 0.\overline{15} \\ 100N &= 15.\overline{15} \\ -N &= -0.\overline{15} \\ \hline 99N &= 15 \\ N &= \frac{15}{99} = \frac{5}{33} \end{aligned}$$

a. $0.\overline{2}$

b. $0.\overline{123}$

c. $1.\overline{35}$

d. $0.\overline{9}$

56. Find the error in the following argument. Assume that $a = b \neq 0$.

$$\begin{aligned} a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a + b)(a - b) &= b(a - b) \\ a + b &= b \\ b + b &= b \text{ (since } a = b) \\ 2b &= b \\ 2 &= 1 \text{ (since } b \neq 0) \end{aligned}$$

57. Solve

a. $\dfrac{w - c}{w - d} = \dfrac{c^2}{d^2}$ $\quad$ for w

b. $a^2 = \dfrac{a + c}{x} + c^2$ $\quad$ for x

c. $(a - y)(y + b) - c(y + c) = (c - y)(y + c) + ab$ for y

58. Solve for y.

a. $y + \dfrac{c}{y-3} = 3 + \dfrac{c}{y-3}$

b. $y + \dfrac{c}{y-3} = -3 + \dfrac{c}{y-3}$

59. The golden ratio is given by

$$T = \frac{1 + \sqrt{5}}{2}$$

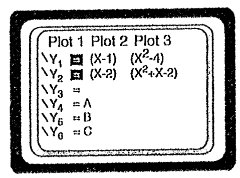
(See Chapter 1, Review Exercise 78.) Show that

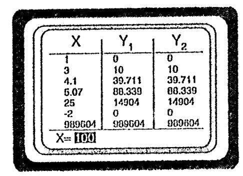
$$T = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{T}}}$$

60. Determine if the equation is an identity experimentally by setting up a TABLE in your graphing calculator and entering a variety of values for x.

$$(x^2 + 5x + 6)(x - 3) = (x^2 - x - 6)(x + 3)$$

Example for Exercise 60:





61. *Mathematics in Writing:* Set up a table like the one in Exercise 60, but pick a Y_1 and Y_2 so that $Y_1 = Y_2$ is a conditional equation. Explain in a brief paragraph how the tables differ and what the difference tells you about the equations.

2.2 Applications: From Words to Algebra

Many applied problems lead to linear equations. The challenge of applied problems is translating words into appropriate algebraic forms.

The steps listed here can guide you in solving word problems.

Step 1. Read the problem through the first time to get a general idea of what is being asked.

Step 2. Read the problem a second time to recognize what may be important in determining that which is to be found.

Step 3. If possible, estimate the solution to this problem, and then compare this estimate with your final answer.

Step 4. Let some algebraic symbol denote the quantity to be found.

Step 5. If possible, represent other quantities in the problem in terms of the algebraic symbol designated in *Step 4*.

Step 6. Find various relationships (equations or inequalities) in the problem.

Step 7. Use relationships established in *Step 6* to find the solution to the problem.

Step 8. Verify that your answer is, indeed, the solution to the problem.

The words and phrases in Table 1 may prove helpful in translating a word problem into an algebraic expression that can be solved.

EXAMPLE 1 PRICES AND DISCOUNTS

If you pay \$66 for a car radio after receiving a 25% discount, what was the price of the radio before the discount?

SOLUTION

Let p = the price of the radio (in dollars) before the discount. Then

$$0.25p = \text{the amount discounted}$$

and the price of the radio after the discount is given by

$$p - 0.25p$$

Hence

$$\begin{aligned} p - 0.25p &= 66 \\ 0.75p &= 66 \\ p &= \frac{66}{0.75} = 88 \end{aligned}$$

The price of the radio was \$88 before the discount.

TABLE 1 **Translation of Words into Algebraic Expressions**

Word or Phrase	Algebraic Symbol	Example	Algebraic Expression
Sum	$+$	Sum of two numbers	$a + b$
Difference	$-$	Difference of two numbers Difference of a number and 3	$a - b$ $x - 3$
Product	$\times$ or $\cdot$	Product of two numbers	$a \cdot b$, $(a)(b)$, or ab
Quotient	$\div$ or /	Quotient of two numbers	$\frac{a}{b}$, a/b, or $a \div b$
Exceeds More than More of		a exceeds b by 3 a is 3 more than b There are 3 more of a than of b.	$a = b + 3$ or $a - 3 = b$
Twice		Twice a number Twice the difference of x and 3 3 more than twice a number 3 less than twice a number	$2x$ $2(x - 3)$ $2x + 3$ $2x - 3$
Is or equals	$=$	The sum of a number and 3 is 15.	$x + 3 = 15$

Coin Problems

When interpreting coin problems, always distinguish between the *number* of coins and the *value* of the coins. You may also find it helpful to use a chart, as in the following example:

EXAMPLE 2 COINS

A purse contains \$3.20 in quarters and dimes. If there are 3 more quarters than dimes, how many coins of each type are there?

SOLUTION

In this problem, we may let the unknown represent either the number of quarters or the number of dimes. Let

$$q = \text{the number of quarters}$$

Then

$$q - 3 = \text{the number of dimes}$$

since there are 3 more quarters than dimes.

Note that the number of coins times the value of each coin in cents is equal to the total value in cents using that particular coin.

	Number of coins ×	Value of each coin in cents	= Total value in cents using that coin
Quarters	q	25	$25q$
Dimes	$q - 3$	10	$10(q - 3)$

We know that

$$\text{total value} = (\text{value of quarters}) + (\text{value of dimes})$$

$$\begin{aligned} 320 &= 25q + 10(q - 3) \\ 320 &= 25q + 10q - 30 \\ 350 &= 35q \\ 10 &= q \end{aligned}$$

Then

$$\begin{aligned} q &= \text{number of quarters} = 10 \\ q - 3 &= \text{number of dimes} = 7 \end{aligned}$$

Now verify that the total value of all the coins is \$3.20.

Simple Interest

Interest is the fee charged for borrowing money. In this section we will deal only with simple interest, which assumes the fee to be a fixed percentage r of the amount borrowed. We call the amount borrowed the **principal** and denote it by P.

If the principal P is borrowed at a simple annual interest rate r, then the interest due at the end of each year is Pr, and the total interest I due at the end of t years is

$$I = Prt$$

Consequently, if S is the total amount owed at the end of t years, then

$$S = P + I = P + Prt$$

since both the principal and interest are to be repaid. Thus, the basic formulas for simple interest calculations are

$$I = Prt$$
$$S = P + Prt$$

EXAMPLE 3 SIMPLE INTEREST

A part of $7000 was borrowed at 6% simple annual interest and the remainder at 8%. If the total amount of interest due after 3 years is $1380, how much was borrowed at each rate?

SOLUTION

Let

$$s = \text{the amount borrowed at } 6\%$$

Then

$$7000 - s = \text{the amount borrowed at } 8\%$$

since the total amount is $7000. We can display the information in table form using the equation $I = Prt$.

	P	×	r	×	t	=	Interest
6% Portion	s		0.06		3		$0.18s$
8% Portion	$7000 - s$		0.08		3		$0.24(7000 - s)$

Note that we write the rate r in its decimal form, so that $6\% = 0.06$ and $8\% = 0.08$.

Since the total interest of \$1380 is the sum of the interest from the two portions, we have

$$
\begin{aligned}
1380 &= 0.18s + 0.24(7000 - s) \\
1380 &= 0.18s + 1680 - 0.24s \\
0.06s &= 300 \\
s &= 5000
\end{aligned}
$$

We conclude that \$5000 was borrowed at 6% and \$2000 was borrowed at 8%.

Distance Problems (Uniform Motion)

Here is the key to the solution of distance problems.

$$\text{Distance} = (\text{Rate})(\text{Time})$$

or

$$d = r \cdot t$$

The relationships that permit you to write an equation are sometimes obscured by the words. Here are some questions to ask as you set up a distance problem.

1. Are there two distances that are equal? (Will two objects have traveled the same distance? Is the distance on a return trip the same as the distance going?)
2. Is the sum (or difference) of two distances equal to a constant? (When two objects are traveling toward each other, they meet when the sum of the distances traveled by them equals the original distance between them.)

EXAMPLE 4 TRAVEL

Two trains leave New York for Chicago. The first train travels at an average speed of 60 mph. The second train, which departs an hour later, travels at an average speed of 80 mph. How long will it take the second train to overtake the first train?

SOLUTION

Since we are interested in the time the second train travels, we let

t = the number of hours the second train travels

Then

$$t + 1 = \text{the number of hours the first train travels}$$

since the first train departs 1 hour earlier. We display the information in table form using the equation $d = rt$.

	Rate	×	Time	=	Distance
First train	60		$t + 1$		$60(t + 1)$
Second train	80		t		$80t$

At the moment the second train overtakes the first, they must both have traveled the *same* distance. Thus,

$$60(t + 1) = 80t$$
$$60t + 60 = 80t$$
$$60 = 20t$$
$$3 = t$$

It takes the second train 3 hours to catch up with the first train.

Mixture Problems

One type of mixture problem involves mixing varieties of a commodity, say two or more types of coffee, to obtain a mixture with a desired value. If the commodity is measured in pounds, the relationships we need are as follows:

$$(\text{Number of pounds})(\text{Price per pound}) = \text{Value of commodity}$$
$$\text{Sum of weights of all varieties} = \text{Weight of mixture}$$
$$\text{Sum of values of all varieties} = \text{Value of mixture}$$

EXAMPLE 5 MIXTURES

How many pounds of Brazilian coffee worth \$10 per pound must be mixed with 20 pounds of Colombian coffee worth \$8 per pound to produce a mixture worth \$8.40 per pound?

SOLUTION

Let B = number of pounds of Brazilian coffee. We display all the information, using cents in place of dollars.

Type of coffee	Number of pounds	× Price per pound	= Value (in cents)
Brazilian	B	1000	$1000B$
Colombian	20	800	16,000
Mixture	$B + 20$	840	$840(B + 20)$

Note that the weight of the mixture equals the sum of the weights of the Brazilian and Colombian coffees that make up the mixture. Since the value of the mixture is the sum of the values of the two types of coffee,

$$\begin{aligned} \text{value of mixture} &= (\text{value of Brazilian}) + (\text{value of Colombian}) \\ 840(B + 20) &= 1000B + 16{,}000 \\ 840B + 16{,}800 &= 1000B + 16{,}000 \\ 800 &= 160B \\ 5 &= B \end{aligned}$$

We must add 5 pounds of Brazilian coffee to make the required mixture.

Work Problems

Work problems typically involve two or more people or machines working on the same task. The key to these problems is to express the *rate of work per unit of time,* whether an hour, a day, a week, or some other unit. For example, if a machine can do a job in 5 days, then

$$\text{rate of machine} = \frac{1}{5} \text{ job per day}$$

If this machine is used for 2 days, it performs $2(\frac{1}{5}) = \frac{2}{5}$ of the job. In summary:

If a machine (or person) can complete a job in n days, then

$$\text{Rate of machine (or person)} = \frac{1}{n} \text{ Job per day}$$

$$\text{Work done} = (\text{Rate})(\text{Time})$$

EXAMPLE 6 WORK

Using a small mower, at 12 noon a student begins to mow a lawn, a job that would take 9 hours working alone. At 1 P.M. another student, using a tractor, joins the first student and they complete the job together at 3 P.M. How many hours would it take to do the job using only the tractor?

SOLUTION

Let t = number of hours to do the job by tractor alone. The small mower works from 12 noon to 3 P.M., or 3 hours. The tractor is used from 1 P.M. to 3 P.M., or 2 hours.

All the information can be displayed in table form.

	Rate	× Time	= Work done
Small mower	$\frac{1}{9}$	3	$\frac{3}{9} = \frac{1}{3}$
Tractor	$\frac{1}{t}$	2	$\frac{2}{t}$

Since

$$\begin{array}{ccccc} \text{Work done by small mower} & + & \text{Work done by tractor} & = & 1 \text{ Whole job} \\ \frac{1}{3} & + & \frac{2}{t} & = & 1 \end{array}$$

To solve, multiply both sides by the LCD, which is $3t$.

$$\left(\frac{1}{3} + \frac{2}{t}\right)(3t) = 1(3t)$$
$$t + 6 = 3t$$
$$t = 3$$

Thus, by tractor alone, the job can be done in 3 hours.

Formulas

The circumference C of a circle is given by the formula

$$C = 2\pi r$$

where r is the radius of the circle. For every value of r, the formula gives us a value of C. If $r = 20$, we have

$$C = 2\pi(20) = 40\pi$$

It is sometimes convenient to be able to turn a formula around, that is, to be able to solve for a different variable. For example, if we want to express the radius of a circle in terms of the circumference, we have

$$C = 2\pi r$$
$$\frac{C}{2\pi} = \frac{2\pi r}{2\pi} \quad \text{Dividing by } 2\pi$$
$$\frac{C}{2\pi} = r$$

Now, given a value of C, we can determine a value of r.

EXAMPLE 7 MANIPULATION OF FORMULAS

If an amount P is borrowed at the simple annual interest rate r, then the amount S due at the end of t years is

$$S = P + Prt$$

Solve for P.

SOLUTION

$$P + Prt = S$$

$$P(1 + rt) = S \qquad \text{Common factor } P$$

$$P = \frac{S}{1 + rt} \qquad \text{Dividing both sides by } (1 + rt)$$

Graphing Calculator Alert

A formula may be stored as a PROGRAM in your graphing calculator. The program may be written merely to display the formula, or to ask the user for the input values and then evaluate the formula for those particular values. Many formulas are available online, or you could learn to write your own. Consult your owner's manual for details. The owner's manual may be available online.

Exercise Set 2.2

In Exercises 1–3, let n represent the unknown. Translate from words to an algebraic expression or equation.

1. The number of blue chips is 3 more than twice the number of red chips.
2. The number of station wagons on a parking lot is 20 fewer than 3 times the number of sedans.
3. Five less than 6 times a number is 26.

In Exercises 4–41, translate from words to an algebraic problem and solve.

4. Janis is 3 years older than her sister. Thirty years from now the sum of their ages will be 111. Find the current ages of the sisters.
5. John is presently 12 years older than Fred. Four years ago John was twice as old as Fred. How old is each now?
6. The larger of two numbers is 3 more than twice the smaller. If their sum is 18, find the numbers.
7. Find three consecutive integers whose sum is 21.
8. A certain number is 5 less than another number. If their sum is 11, find the two numbers.
9. A resort guarantees that the average temperature over the period Friday, Saturday, and Sunday will be exactly 80°F, or else each guest pays only half price for the facilities. If the temperatures on Friday and Saturday were 90°F and 82°F, respectively, what must the temperature be on Sunday so that the resort does not lose half of its revenue?
10. A patient's temperature was taken at 6 A.M., 12 noon, 3 P.M., and 8 P.M. The first, third, and fourth readings were 102.5°, 101.5°, and

102°F, respectively. The nurse forgot to write down the second reading, but recorded that the average of the four readings was 101.5°F. What was the second temperature reading?

11. A 12-meter long steel beam is to be cut into two pieces so that one piece will be 4 meters longer than the other. How long will each piece be?

12. A rectangular field whose length is 10 meters longer than its width is to be enclosed with exactly 100 meters of fencing material. What are the dimensions of the field?

13. A vending machine contains $3.00 in nickels and dimes. If the number of dimes is 5 more than twice the number of nickels, how many coins of each type are there?

14. A wallet contains $460 in $5, $10, and $20 bills. The number of $5 bills exceeds twice the number of $10 bills by 4, and the number of $20 bills is 6 fewer than the number of $10 bills. How many bills of each type are there?

15. A movie theater charges $7.50 admission for an adult and $5 for a child. If 700 tickets are sold on a particular day and the total revenue received is $4500, how many tickets of each type are sold?

16. A student bought 23-cent, 41-cent, and 80-cent stamps with a total value of $31.50. If the number of 23-cent stamps is 2 more than the number of 41-cent stamps, and the number of 80-cent stamps is 5 more than one-half the number of 41-cent stamps, how many stamps of each denomination did the student obtain?

17. An amateur theater group is converting a classroom to an auditorium for a forthcoming play. The group sells $3, $5, and $6 tickets, and receives exactly $503 from the sale of tickets. If the number of $5 tickets is twice the number of $6 tickets, and the number of $3 tickets is 1 more than 3 times the number of $6 tickets, how many tickets of each type are there?

18. To pay for their child's college education, the parents invested $10,000, part in a certificate of deposit paying 8.5% annual interest, the rest in a mutual fund paying 7% annual interest. The annual income from the certificate of deposit is $200 more than the annual income from the mutual fund. How much money was put into each type of investment?

19. A bicycle store is closing out its entire stock of a certain brand of three-speed and ten-speed models. The profit on a three-speed bicycle is 11% of the sale price, and the profit on a ten-speed model is 22% of the sale price. If the entire stock is sold for $16,000 and the profit on the entire stock is 19%, how much is obtained from the sale of each type of bicycle?

20. A film shop carrying black-and-white film and color film has $4000 in inventory. The profit on black-and-white film is 12%, and the profit on color film is 21%. If all the film is sold, and if the profit on color film is $150 less than the profit on black-and-white film, how much was invested in each type of film?

21. A firm borrowed $12,000 at a simple annual interest rate of 8% for a period of 3 years. At the end of the first year, the firm found that its needs were reduced. The firm returned a portion of the original loan and retained the remainder until the end of the 3-year period. If the total interest paid was $1760, how much was returned at the end of the first year?

22. A finance company lent a certain amount of money to Firm A at 7% annual interest. An amount $100 less than that lent to Firm B at 8%, and an amount $200 more than that lent to Firm A was lent to Firm C at 8.5%. All loans were for one year. If the total annual income is $126.50, how much was lent to each firm?

23. Two trucks leave Philadelphia for Miami. The first truck to leave travels at an average speed of 50 kilometers per hour. The second truck, which leaves 2 hours later, travels at an average speed of 55 kilometers per hour.

How long does it take the second truck to overtake the first truck?

24. Jackie either drives or bicycles from home to school. Her average speed when driving is 36 mph, and her average speed when bicycling is 12 mph. If it takes her $\frac{1}{2}$ hour less to drive to school than to bicycle, how long does it take her to go to school, and how far is the school from her home?

25. Professors Roberts and Jones, who live 676 miles apart, are exchanging houses and jobs for the summer. They start out for their new locations at exactly the same time, and they meet after 6.5 hours of driving. If their average speeds differ by 4 mph, what are their average speeds?

26. Steve leaves school by moped for spring vacation. Forty minutes later his roommate, Frank, notices that Steve forgot to take his camera. So, Frank decides to try to catch up with Steve by car. If Steve's average speed is 25 mph and Frank averages 45 mph, how long does it take Frank to overtake Steve?

27. An express train and a local train start out from the same point at the same time and travel in opposite directions. The express train travels twice as fast as the local train. If after 4 hours they are 480 kilometers apart, what is the average speed of each train?

28. How many pounds of raisins worth \$3 per pound must be mixed with 10 pounds of peanuts worth \$2.40 per pound to produce a mixture worth \$2.80 per pound?

29. How many ounces of Ceylon tea worth \$1.50 per ounce and how many ounces of Formosa tea worth \$2.00 per ounce must be mixed to obtain a mixture of 8 ounces that is worth \$1.85 per ounce?

30. A copper alloy that is 40% copper is to be combined with a copper alloy that is 80% copper to produce 120 kilograms of an alloy that is 70% copper. How many kilograms of each alloy must be used?

31. A vat contains 27 gallons of water and 9 gallons of acetic acid. How many gallons of water must be evaporated if the resulting solution is to be 40% acetic acid?

32. A producer of packaged frozen vegetables wants to market mixed vegetables at \$1.20 per kilogram. How many kilograms of green beans worth \$1.00 per kilogram must be mixed with 100 kilograms of corn worth \$1.30 per kilogram and 90 kilograms of peas worth \$1.40 per kilogram to produce a satisfactory mixture?

33. A certain number is 3 times another. If the difference of their reciprocals is 8, find both numbers.

34. If $\frac{1}{3}$ is subtracted from 3 times the reciprocal of a certain number, the result is $\frac{25}{6}$. Find the number.

35. Computer A can carry out an engineering analysis in 6 hours, but computer B can do the same job in 4 hours. How long does it take to complete the job if both computers work together?

36. Jackie can paint a certain room in 3 hours, Lisa in 4 hours, and Susan in 2 hours. How long does it take to paint the room if they all work together?

37. A senior copy editor together with a junior copy editor can edit a book in 3 days. The junior editor, working alone, would take twice as long to complete the job as the senior editor would require if working alone. How long would it take each editor to complete the job by herself?

38. Hose A can fill a certain vat in 3 hours. After 2 hours of pumping, hose A is turned off. Hose B is then turned on and completes filling the vat in 3 more hours. How long would it take hose B to fill the vat alone?

39. A printing shop starts a job at 10 A.M. on press A. Using this press alone, it would take 8 hours to complete the job. At 2 P.M. press B is also turned on, and both presses together

finish the job at 4 P.M. How long would it take press B to do the job alone?

40. A boat travels 20 kilometers upstream in the same time that it would take the same boat to travel 30 kilometers downstream. If the rate of the stream is 5 kilometers per hour, find the speed of the boat in still water.

41. An airplane flying against the wind travels 300 miles in the same time that it would take the same plane, flying the same speed, to travel 400 miles with the wind. If the wind speed is 20 mph, find the speed of the airplane in still air.

In Exercises 42–51 solve for the indicated variable in terms of the remaining variables.

42. $A = Pr$ for r

43. $C = 2\pi r$ for r

44. $V = \frac{1}{3}\pi r^2 h$ for h

45. $F = \frac{9}{5}C + 32$ for C

46. $S = \frac{1}{2}gt^2 + vt$ for v

47. $A = \frac{1}{2}h(b + b')$ for b

48. $A = P(1 + rt)$ for r

49. $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ for f_2

50. $a = \frac{v_1 - v_0}{t}$ for v_0

51. $S = \frac{a - rL}{L - r}$ for L

52. Translate the following from words to an algebraic expression or equation, denoting the unknown by n.
   a. The express train travels 5 mph faster than the local train.
   b. The length of a rectangle is 7 inches more than its width.
   c. the area of a triangle, if the altitude is twice the base
   d. the sum of 3 consecutive even numbers
   e. 15% of the amount by which a number exceeds 10,000

53. If r and s represent two numbers, write the following:
   a. twice the sum of the two numbers
   b. 5% of the difference between the two numbers
   c. 5 less than twice the second number
   d. the ratio of the first to the second number
   e. the sum of the squares of the two numbers
   f. the average of the two numbers
   g. 6 times the first number less 4 times the second number

54. Write formulas for each of the following:
   a. the charge in cents for a telephone call between two cities lasting n minutes, n greater than 3, if the charge for the first 3 minutes is \$1.20 and each additional minute costs 33 cents
   b. the taxi fare for m miles, if the initial charge is \$2.50 and the driver charges 70 cents for every $\frac{1}{5}$ mile traveled
   c. the amount in an account at the end of a year, if simple interest is paid at the rate of 16%, and the account contains d dollars at the beginning of the year
   d. the fine a company paid for dumping acid into the Mississippi River for d days, if the U.S. Environmental Protection Agency fined the company \$150,000 plus \$1000 per day until the company complied with the federal water pollution regulations.

55. Find three consecutive even numbers such that twice the first plus 3 times the second is 4 times the third.

56. When exercising, Mary walks a distance to warm up, jogs $3\frac{1}{2}$ times as far as she walks, and sprints $3\frac{1}{3}$ times as far as she jogs. If she covers 4171 meters, find the distances that she walked, jogged, and sprinted.

57. A 10-quart radiator has 30% antifreeze. How much of the fluid should be drained and replaced with pure antifreeze to double the strength of the mixture?

58. There are two identical beakers in a chemistry laboratory, both filled to the same level. One contains sulfuric acid and the other contains water. First, one spoon of acid is put into the beaker with the water and mixed thoroughly. Then one spoon of this mixture is put back into the beaker with the acid. Is there more water in the acid or more acid in the water?

59. Two bicyclists leave cities A and B at the same time, heading toward each other. Their speeds are 20 mph and 30 mph, respectively. The distance between these cities is 100 miles. Simultaneously, a bird leaves city A, heading toward B, traveling at 40 mph. When it meets the bicyclist who left from B, it turns around and heads back toward A. When it subsequently meets the bicyclist who came from A, it turns around and heads back toward B, and so on. Find the total distance the bird will have flown by the time the two bicyclists meet.

60. To determine the number of deer in a forest, a conservationist catches 225 deer, tags them and then releases them. A week later, 102 deer are caught and, of those, 15 are found to be tagged. Assuming that the proportion of tagged deer in the second sample was the same as the proportion of all tagged deer in the total population, estimate the number of deer in the forest.

61. In a Tour de France bicycle race, Stefan averaged 20 mph for the first third of the race and 35 mph for the remainder. Enrique maintained a constant speed of 30 mph throughout the race. Of these two, who finished first?

In Exercises 62–65, solve for the indicated variable.

62. $I = \dfrac{E}{r + \dfrac{R}{n}}$ for n

63. $Wf = \left(\dfrac{W}{k} - 1\right)\left(\dfrac{1}{k}\right)$ for W

64. $W = \dfrac{2PR}{R - r}$ for r

65. $\dfrac{E}{c} = \dfrac{R + r}{r}$ for r

2.3 The Quadratic Equation

We now turn our attention to equations involving second-degree polynomials. A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

where a, b, and c are real numbers. In this section we will explore techniques for solving this important class of equations. We will also show that there are several kinds of equations that can be transformed into quadratic equations and then solved.

Solving by Factoring

If we can factor the left-hand side of the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

into two linear factors, then we can solve the equation. For example, the quadratic equation

9. -1 10. 2 11. 25

12. $-\frac{7}{3}$ 13. b 14. $-2.2, 5$

15. 14, 6

16. $2xy + 3x + 4y + 1$

17. $3a^3 + 5a^2 + 3a + 10$

18. $4a^2b(2ab^4 - 3a^3b + 4)$

19. $(2 - 3x)(2 + 3x)$

20. $\frac{6m^5}{n^2}$ 21. $\frac{1-x}{x+1}$

22. $4x^2(x + 1)(x - 1)(x - 2)$

23. $\frac{11x - 15}{3(x^2 - 9)}$ 24. $\frac{2}{x+1}$ 25. $\frac{1}{x^{17}}$

26. y^{n+1} 27. -1 28. $\frac{4a^4}{b^2}$

29. 0 30. $32 - 10\sqrt{7}$ 31. $-\frac{11\sqrt{xy}}{4}$

32. $x \le 2$ 33. $-1 + 0i$ 34. $16 - 11i$

35. $\frac{8}{5} + \frac{9}{5}i$

Chapter 2

Exercise Set 2.1

1. T 3. T 5. -2

7. $-\frac{2}{3}$ 9. 6 11. $-\frac{4}{3}$

13. $\frac{3}{2}$ 15. $-\frac{10}{3}$ 17. -2

19. 5 21. $-\frac{7}{2}$ 23. 1

25. $\frac{8}{5-k}$ 27. $\frac{6+k}{5}$ 29. $\frac{10}{3}$

31. 1 33. 4 35. 4

37. 12 39. 2 41. $\frac{12}{7}$

43. none 45. I 47. C

49. T 51. F 53. T

55. a. $\frac{2}{9}$ b. $\frac{41}{333}$ c. $\frac{134}{99}$ d. 1

57. a. $w = \frac{cd}{c+d}$ b. $x = \frac{1}{a-c}$ c. $y = \frac{2c^2}{a-b-c}$

Exercise Set 2.2

1. $2n + 3$ 3. $6n - 5 = 26$ 5. 16, 28

7. 6, 7, 8 9. 68°

11. 4 meters and 8 meters

13. 10 nickels, 25 dimes

15. 300 children, 400 adults

17. 61 three-dollar tickets, 40 five-dollar tickets, 20 six-dollar tickets

19. \$11,636.36 on 10-speeds, \$4363.64 on 3-speeds

21. \$7000 23. 20 hours

25. 50 miles per hour and 54 miles per hour

27. 40 kilometers per hour, 80 kilometers per hour

29. Ceylon: 2.4 ounces, Formosa: 5.6 ounces

31. 13.5 gallons 33. $\frac{1}{12}, \frac{1}{4}; -\frac{1}{4}, -\frac{1}{12}$

35. 2.4 hours 37. $4\frac{1}{2}$ days, 9 days

39. 8 hours 41. 140 miles per hour

43. $\frac{C}{2\pi}$ 45. $\frac{5(F-32)}{9}$ 47. $\frac{2A}{h} - b'$

49. $\frac{ff_1}{f_1 - f}$ 51. $\frac{a + Sr}{r + S}$

53. a. $2(r + s)$ b. $0.05|r - s|$ c. $2s - 5$
d. $\frac{r}{s}$ e. $r^2 + s^2$ f. $\frac{r+s}{2}$
g. $6r - 4s$

55. 10, 12, 14 57. $\frac{30}{7}$ quarts 59. 80 miles

61. Enrique 63. $W = \frac{k}{1 - fk^2}$ 65. $r = \frac{cR}{E - c}$

Exercise Set 2.3

1. 1, 2 3. 1, -2 5. $-2, -4$

7. 0, 4 9. $\frac{1}{2}, 2$ 11. ± 2

13. $\frac{1}{3}, \frac{1}{2}$ 15. ± 3 17. $\pm\sqrt{5}$

19. $-\frac{5}{2} \pm \sqrt{2}$ 21. $\frac{5 \pm 2\sqrt{2}}{3}$ 23. $\pm\frac{8i}{3}$