SECTION 1.1

1. Responses will vary. They may include technical geometric terms such as perpendicular, parallel, right angle or more generic terms such as 'the same' and 'across and down'. A good way to asses the quality of the directions is to give them to someone else and see what figure is produced by following the directions.

2. Responses will vary. Some students may produce a technical definition similar to: the shape is a closed planar figure formed by joining four congruent line segments with four right angles. Others may add redundant conditions such as including the fact that opposite sides are parallel to the previous description. Still other students may use less formal language by saying that a square is a box with four sides that are the same.

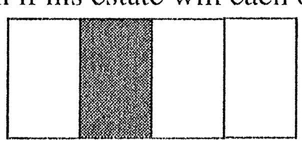
3. Responses will vary among students. Some may choose to use words describing sets of numbers: rational, real, etc. Others may choose to use geometric terms. And yet other students may choose to use terms associated with computations.

4. Responses will vary among students. Some may choose to use the symbols for computations, others symbols for numbers such as π or e. Some may use more esoteric symbols that they have encountered such as Σ.

5. Responses will vary among students. Possibilities include: **XXX** XX; 0.6, 6/10.

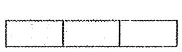
6. Responses will vary among students. Possibilities include: six groups of 4 objects, $20+4$, $2^3\times 3$, 96/4, $\sqrt{576}$, $10+10+4$.

7. Responses will vary. Possibilities include:
Verbal: Michael has 4 children, and he wants to share his estate equally among them. What portion if his estate will each one receive?
Visual:

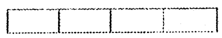
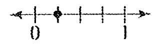


Numerical: 1/4 or 0.25...
Graphic:

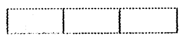
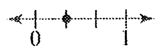
8.

Visual	Numerical	Graphical
	$\frac{2}{3}$	

9.

Visual	Numerical	Graphical
	$\frac{1}{4}$	

10.

Visual	Numerical	Graphical
	$\frac{1}{3}$	

11. Responses will vary. One possibility is: Cut the pizza in half along a diameter. Cut the pizza in half along a diameter again, making sure the two cuts are perpendicular to one another. Take 1 of the pieces of pizza that has been created.

12. Responses will vary. A possibility is: The states with the smallest land area are growing at a faster rate than the states with the largest land area.

13. Responses will vary among students. The message conveyed by the graph is ambiguous. Although better than 1 in 10 women do not conduct self-exams, over half (38% + 4% + 11%) do follow the once per month recommendation. Are we pleased that over half of the women do work for early detection and thus enhance survival chances or are we displeased that a significant proportion do not follow the recommendation?

14. **One self-exam per month** is the most frequent occurrence, performed by 38% of the women responding to the survey.

15. **Fifty three percent** (38% + 4% + 11%) of the women responding to the survey perform self-exams at least once a month.

16. Responses will vary among students. One possibility is a table.

Breast Self-exam Frequency

Never	Less than 1/yr	1-3/yr	4-6/yr	1/month	2-4/month	2-7/week	Don't know
12%	7%	14%	11%	38%	11%	4%	3%

17. a. The value in B2 is 10% of the value in A2. So **B2 = .10 * A2.**
    b. C2 can be directly calculated from A2 (**C2 = .90 * A2**) or by subtracting B2 from A2.

18. A **fraction** calculator would be used. A four-function calculator with key-strokes 3 ÷ 8 + 4 ÷ 5 = would give a decimal result.

19. A **programable scientific** calculator would be appropriate. Another approach would be to use a graphing calculator to graph the equation and then use the trace function.

20. A **graphing calculator** would serve the purpose.

21. Discussion of the usefulness of the software will vary with the experiences and perspectives of the students. One important characteristic of GES is that it permits many trials of a situation rapidly. Thus conjectures may be formulated or counter-examples found. The software enables students to

obtain tables of measures easily and rapidly. GES emphasizes the inductive component of mathematics in non-arithmetic areas.

22. The selection will vary among students. In their responses the students should, perhaps, include the idea that the choice of presentation is related not only to what is being presented but also is dependent upon the characteristics of the audience. To many, the equation may be meaningless. The tabular presentation is the most obvious.

23. Responses will vary. Possibilities include:
    a. What is the most common response of families to migraine sufferers?
    b. The graph does not convey characteristics of migraine sufferers themselves nor the number of sufferers.
    c. This is a matter of opinion. Migraine sufferers have to cope with the headaches, families have to cope with persons with migraines.
    d. Families may have placed themselves in more than one category.

24. The responses will vary among students. Some possible considerations are: the extent to which each is considered to be open to interpretation, the degrees to which 'new ground' can be broken in each area, the extent of relation to reality, the part played by creativity.

25. The results of the exploration suggest that the segment joining the midpoints of 2 sides of a triangle has a length equal to one half of the length of the remaining side.

26. a. The solution requires that the spreadsheet be continued until the value of the car is less than \$10,000. Each entry in the B column after year 1 is .8 of the value immediately above.
    So: A3 = 1, B3 = .8 * B2 = 24,000; A4 = 2, B4 = 0.8 * B3 = 19,200; A5 = 3, B5 = 15,360; A6 = 4, B6 = 12,288; A7 = 5, B7 = 9830.40
    b. The value is less than \$10,000 in **year 5** and the spreadsheet has **7 rows.**

27. All of the figures in the first collection have a right angle at the marked vertex. None of the figures in the second collection has a right angle at the marked vertex. The first collection of figures illustrates "right angleness."

28.

Number of hours	cost: plan A	cost: plan B
5	90	140
6	108	148
7	126	156
8	144	164
9	162	172
10	**180**	**180**
11	198	188

The plans are equal in cost for ten hours of play. Plan B is more economical for **11 or more** hours of play.

29. The estimates will vary among students. Geometry exploration software produces the result that EFGH is one fifth of ABCD.

30. Responses will vary among students.

31.

Hens	Rabbits	Feet
50	0	50
40	10	120
30	20	140

Thus the farmer has **30 hens and 20 rabbits**. Note that in employing a spreadsheet it is often useful to begin with rather coarse increments to establish the range containing the required value. One may then use smaller increments. In this case the broader increment contained the exact answer.

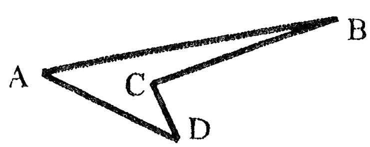
32. Responses will vary among students. One possibility is that students will recognize that the graph indicates an optimum study time. This optimum, most probably, varies among individuals. Perhaps students should experiment to determine their own optimum study times for mathematics. This is, most probably, not a 'night before' study session.

33. Thales discovered that angles inscribed in semicircles are right angles.

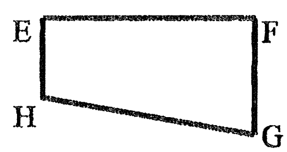
SECTION 1.2

1. The four examples show that adding 0 to a number results in the number. One might generalize from these examples that the sum of 0 and any number is the original number.

2. Sixteen is an even number which does not end in 2.

3. 

ABCD is not a square.

4. 

EFGH is not a rectangle.

5. $20 \div 4 = 5$; 5 is not an even number.

6. a. An **arithmetic sequence** has consecutive terms with a common difference.
   b. Consecutive terms of a **geometric sequence** have a common ratio.

7. 2, 6, 10, 14, 18, **<u>22</u>**, **<u>26</u>**, **<u>30</u>**: **arithmetic ($d = 4$).**

8. 1, 2, 4, 8, 16, **<u>32</u>**, **<u>64</u>**, **<u>128</u>**: **geometric, (r=2)**

9. 1 ,4, 9, 16, 25, **<u>36</u>, <u>49</u>, <u>64</u>: neither** (sequence is squares of consecutive integers).

10. 2, 4, 6, 10, 16, **<u>26</u>, <u>42</u>, <u>68</u>: neither** (Fibonacci-like sequence).

11. 1, 3, 7, 15, 31, **63, 127, 255: neither** (to produce next term double the current term and add 1).

12. 81, 27, 9, 3, **1, 1/3, 1/9: geometric, (r = 1/3).**

13. 1, 2, 4, 7, **11, 16, 22: neither** (the differences between consecutive terms are the consecutive natural numbers).

14. 1, 5, 9, 13, **17, 21, 25: arithmetic (d = 4).**

15. 25, 21, 17, 13, **9, 5, 1: arithmetic, (d = - 4).**

16. Responses will vary. Possibilities include any number ending in 1 or 9 as a counterexample. For example, 11 squared is 121, a number that doesn't end in 4, 5, 6, or 9.

17. Responses will vary. They may include the ideas that:
    a. Conclusions are arrived at through inductive reasoning by assuming that some common aspect of several specific instances applies to the class of similar cases. For example, because GES shows that the sum of the angles of each of several triangles sum to 180°, we conclude that the angles of all triangles sum to 180°. The conclusion could be shown to be false by a *single specific* counterexample.
    b. Conclusions arrived at deductively begin with some general statement accepted as true either by agreement or by prior deductive proof (obviously the entire process of deduction rests upon general statements true by agreement). These statements include requirements that, if met by a more specific situation, allow the conclusion of the general truth to be applied to the more specific situation. For example, we agree that if the sun shines tomorrow then the game will be played. Upon arising we agree that the sun is shining. We can conclude that the game will be played. A conclusion based on deductive reasoning may be challenged by attacking the *form* of the argument rather than the specific content.

18. A car driving 90 mph on Interstate 75 in Georgia would not be speeding. The negation is false.

19. Every square is not a rectangle. The negation is false.

20. Every rectangle is not a square. The negation is true.

21. Hypothesis: an even number is multiplied by an odd number; conclusion: the product is an even number.

22. Hypothesis: a figure has four connected sides; conclusion: it is a quadrilateral.

23. Hypothesis: a product is in high demand; conclusion: the price will increase.

24. Hypothesis: the square of a number is even; conclusion: the number is even.

25. **Inductive reasoning** based on the calculator display suggests the numbers form a geometric sequence with $r = 1/2$. Therefore, continuing the pattern based on this conclusion, a prediction of 1 is reasonable.

26. A statement of the form 'If p, then q.' is called a conditional or an *implication*. Logically, an implication is true in all cases except that in which the hypothesis is true and the conclusion is false. Jack's statement "If the temperature is below 50°, then I won't play tennis." is an implication.
    a. Because the general conditions of both hypothesis and conclusion of Jack's original conditional are supported in this particular situation, this case indicates Jack **told the truth** in his original conditional statement.
    b. Because the hypothesis, but not the conclusion, is supported, this case indicates Jack **did not tell the truth** in his original statement.
    c. This specific case does not meet the conditions set forth in the hypothesis of Jack's general conditional. But **Jack's implication is true, and he did not lie.**
    d. This case is the same as c.

27. Recall that $p \rightarrow q$ is false only when p is true and q is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement is false if Serena wins the second set and loses the match.

28. The statement is "If inflation remains constant (p), then salaries will increase (q)." would be true if p and q both were true, if p and q both were false, or if p were false and q were true. So examples are: Inflation remains constant and salaries do increase.; Inflation did not remain constant and salaries did not increase.; and Inflation did not remain constant and salaries did increase.

29. A right triangle has one right angle and two acute angles. The conjunction is true because both the original statements are true.

30. A rectangle has four right angles and four congruent sides. The conjunction is false because the second statement is false.

31. The number 9 is either prime or odd. The disjunction is true because the second statement is true.

32. Either 6 is a multiple of 3 or it is a multiple of 12. The disjunction is true because the first statement is true.

33. If a triangle has three congruent sides, then it has three congruent angles. If a triangle has three congruent angles, then it has three congruent sides.

34. If a number is divisible by 10, the number ends in zero. If a number ends in zero, it is divisible by 10.

35, 37, 38 are examples of **denying the conclusion.**

36, 39 are examples of **affirming the hypothesis**.

40. This is an example of **inductive** reasoning.

41. This is an example of **deductive** reasoning.

42. This is an example of **invalid deductive reasoning.** The accepted conditional is 'All triangles have three sides.' (If a geometric figure is a triangle, then that figure has three sides.) The particular condition is 'ABC is a triangle,' which affirms the hypothesis. But the conclusion of the conditional is not correctly applied to the specific instance.

43. This is an example of **inductive** reasoning.

44. This is a **valid** example of deductive reasoning by affirming the hypothesis.

45. This reasoning is **invalid.** It is an example of **assuming the converse.**

46. This example is **invalid.** It is an example of **assuming the converse.**

47. This example is **invalid.** It is an example of **assuming the inverse**.

48. If the square of a number is positive, then the number is positive. (False—the number could be negative.)

49. If a number is not positive, then its square is not positive. (False—the number could be negative.)

50. If the square of a number is not positive, then the number is not positive. (True)

51. (a) If a figure is a rhombus, then it has four congruent sides. (b) If a figure has four congruent sides, then it is a rhombus.

52. No, it could be a non-square rhombus.

53. No, it could be a non-square rectangle.

54. The product will not be positive if one number is positive and one number is negative.

55. The conjunction is false because $p \wedge \sim q$ can only be true if both p and $\sim q$ are true. But we know that q is true, and it is not possible for q and $\sim q$ to be true simultaneously.

56. The disjunction is true because $p \vee \sim q$ is true if either p or $\sim q$ is true. We know that p is true, so the disjunction is true.

57. The biconditional statement will be false. I order for $p \rightarrow q$ to be true, both $p \rightarrow q$ and $q \rightarrow p$ must be true.

58.

p	q	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$q \rightarrow \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

59.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow q$	$\sim q \rightarrow p$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

60. The examples suggest that the product 1... 11×12 will be of the form 13…32 with the number of threes one less than the number of ones. Thus $111111 \times 12 =$ **1333332**. A calculator check shows this to be correct.

61. The product in the first column is 1 more than the product in the second column. Then in the next pair, the product in the first column is 1 less than the product in the second column. This pattern continues with other numbers in the sequence:

1 x 3	3	1 x 2	2
1 x 5	5	2 x 3	6
2 x 8	16	3 x 5	15
3 x 13	39	5 x 8	40
5 x 21	105	8 x 13	104
8 x 34	272	13 x 21	273
13 x 55	715	21 x 34	714

62. 65 x 65 = **42**25; 75 x 75 = **56**25; 85 x 85 = **72**25. The last two digits are always 25 (5 x 5). The first two digits are the tens digit of the factors multiplied by the next higher number.

63. Responses will vary. Possibilities include: Assume that the conditional statement "If a quadrilateral is a rhombus, then the sides of that quadrilateral are all the same length." is true (It is.). Now, let geometric figure ABCD be a rhombus. By affirming the hypothesis, we can conclude the sides of ABCD are the same length.

64. Jennifer employed inductive reasoning. Indeed, the numbers 4, 24, 44, and 64 all end in 4 and are evenly divisible by four. Since each of these numbers end in 4, Jennifer generalized that all numbers ending in 4 are evenly divisible by 4. Jennifer's **generalization is false.** The number 14 serves as a counterexample.

65. Responses will vary. A possibility is the set: $3^2 = 9$, $17^2 = 289$, $21^2 = 441$.

66. Responses will vary. Possibilities include: The keystrokes 6, +, 3, =, =, =, = produce the numbers 9, 12, 15, 18. These form an arithmetic sequence. The keystrokes 6, $\times$ 2, =, =, =, = produce 12, 72, 432, 2592. These form a geometric sequence.

67. a. The conclusion is that figure ABCD is a rectangle. We have affirmed the hypothesis.
    b. The conclusion is that figure ABCD is a polygon, again by affirming the hypothesis.
    c. The conclusion is that 15 is an odd number by the same reasoning as in a and b.

68. The converse statement.

69. The inverse statement.

70, 71. **Both generalizations follow** from the examples given. However, in exercise 71 the statement can be shown to be **incorrect** with the counterexample $5^2 - 1$. Now, for the generalization that the square of an odd number -1 is divisible by 8, consider the following argument: let O be the odd number. Then $O = E + 1$, E an even number. And $O^2 - 1 = (E + 1)^2 - 1 = E^2 + 2E$. Now, an even number has a factor of 2 so let $E = 2X$. Thus $E^2 + 2E = (2X)(2X) + 2(2X) = 4X(X) + 4X = 4X(X + 1)$. Now, if X is even we have $X = 2Y$ and $4X(X + 1) = 8Y(X + 1)$, an expression evenly divisible by 8. If X is odd, then $X + 1$ is even and a similar argument shows that $4X(X + 1)$ is divisible by 8. So the statement in exercise 70 **is true** and no counterexample will be found by the students.

72. The argument is **invalid**; it assumes the converse.

73. The argument is a **valid** application of denying the conclusion.

74. **is a valid** application of rule B, denying the conclusion.

75. **is a valid** example of rule A, affirming the hypothesis.

76. The reasoning **is invalid.** The error is assuming the inverse.

77. If two consecutive terms of a geometric sequence are 8 and 12, then $r = 12/8 = 1.5$ and each term can be calculated from the preceding term by multiplying by 1.5. So the sequence containing 8 and 12 would continue 18, 27, ... **Kevin is correct.**

78. Responses will vary. A possibility is: If we accept the conditional: If an animal is a dog, then that animal can fly, then 'Mick is a dog' affirms the hypothesis and we can conclude 'Mick can fly'.

79. Responses will vary among students. Some responses might include: If a person works for Sleezy, then that person will make a fortune; If you work hard, then you will get promoted fast; If you are trained by Sleezy, then you will be smart.

80. It appears that if one lists the odd integers, then the sum of the first 2 odd integers is 2^2, the sum of the first 3 is 3^2, the sum of the first 4 would be 4^2, and so on. Since $1 + 3 + 5 + 7 + 9 + 11$ is the sum of the first 6 odd integers, if the pattern holds, the sum will be $6^2 = \mathbf{36}$. This is the case. Generalizing, we predict that the sum $1 + 3 + 5 + \cdots + y$ is $\mathbf{n^2}$, where n is the number of odd integers being added.

81. Responses will vary. They may include the ideas: The ad promotes the following conditionals: 'If you want to feel your best, then take one Vigorous Vitamin each day' and 'If a person takes Vigorous Vitamins, then that person cares about her health'. Frieda mistakenly has assumed the converse of the first ad statement and has assumed the inverse of the second ad statement. Although Frieda has misinterpreted the ad, she has most probably reached the conclusions hoped for by the ad maker.

82. Responses will vary. A possibility is: Experimentation shows that 0 lines result in 1 triangle, 1 line in 3 triangles, 2 lines in 6 triangles, 3 lines in 10 triangles. The number of lines are in an arithmetic sequence with the first term 0 and the constant difference of 1. The number of triangles are in a sequence with the first term 1 and succeeding terms obtained by adding 2 to get the second term, adding 3 to the second term to get the third, adding 4 to the third to get the fourth, and so on. Inductive reasoning.

83. Heather will weigh 137 pounds. $150-(4+6\times 1.5)=137$.

84. Responses will vary among groups of students.

85. Responses will vary among students. An argument might be something like: The bugs are assuming the conditional that if a person or animal slides down the slide, then that person or animal will become ensnared in the web. Some small animal does slide down. Thus the animal is ensnared. The bugs did pull it off. The final conclusion is that the bugs will eat like kings.

86. Responses will vary among students. The ad relies on deductive reasoning. The primary conditional the ad is attempting to convince pet owners is true is: If Waltham is on the label, then the pet food is exceptionally good. They 'support' their argument by stating that Waltham employs vets and animal nutritionists implying, but not stating, that if scientists, vets, and animal nutritionists are employed then the resulting product will be exceptional.

87. First, use the complete diagonal to get the magic number of 34. Now, column 3 is missing one number which must make the sum of that column 34. The number is 11. Continuing in the same manner we find that row 2 is missing a 5, column 1 must have a 9, the second diagonal needs a 6. Now put a 12 in the column 4 blank and 15 in the row 4 blank. Complete the perfect square by placing a 3 in row 1. Now all rows, columns, and diagonals add to 34.

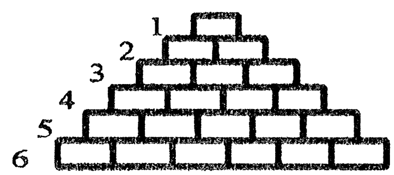
16	**3**	2	13
5	10	**11**	8
9	**6**	7	**12**
4	**15**	14	1

SECTION 1.3

1. Responses will vary among students. Can you do this mentally? With paper/pencil? Need a calculator?

2. Responses will vary. Students might consider that a problem, by definition, is a situation that does not yield to an algorithmic solution. If there were an algorithm for problem solving there would be no problems.

3. a. The diagram shows that the numbers of boxes in consecutive rows form an arithmetic sequence beginning with 1 and with a common difference of 1. One could extend the sequence, adding all the terms, until the sum is 21.
   $1+2+3=6$. $1+2+3+4=10$. $1+2+3+4+5=15$. $1+2+3+4+5+6=21$.
   Thus there are **6** boxes in the bottom row.
   b. The diagram shows that when the sixth row is drawn, the number of boxes totals to 21 and the last row contains 6 boxes.



4. Responses will vary. Some possible combinations are: 4 quarters; 3 quarters, 2 dimes, and 1 nickel; 3 quarters, 1 dime, and 3 nickels.

5. This problem has **one solution: 2 quarters, 2 dimes, and 1 nickel.** One might argue: since there must be 5 coins, none larger than a quarter, there must be at least 2 quarters because no combination of 4 or 5 nickels and dimes total to 75 cents. There cannot be 3 quarters because the total would be attained in 3 coins. So there must be 2 quarters. The remaining 3 coins must total 25 cents and this can be done in only one way: 2 dimes and 1 nickel.

6. The four key stages in the problem-solving process are: **understand the problem, develop a plan, implement the plan, look back over your work.**

7. The responses will vary among students. Possibilities include: make a table.

8. The responses will vary among students. Possibilities include: draw a diagram.

9. The responses will vary among students. Possibilities include: make a table.

10. The responses will vary among students. Possibilities include: making a list.

11. The responses will vary among students. Possibilities include: use reasoning to match and eliminate possible matches.

12. The responses will vary among students. Possibilities include: writing an equation.

13. Answers will vary among students. Possibilities include:
    a. Estimate to the next largest number of units of paint, probably gallons.
    b. Estimate to the nearest second.
    c. Estimate to the nearest $100 dollars.

    In none of these situations do you need an exact answer. You are not asked to reimburse a painter for materials, to determine the winner of the race, nor to pay for the vacation.

14. Responses will vary among students. Among the considerations are: type of phone, auxiliary equipment, basic time package, time spent in roaming areas.

15. The responses will vary among students. The problem could be solved by guess-check-revise. Since the room is rectangular and has a perimeter of 44 ft, the sum of the length and width is 22 ft. If the room were 1 ft by 21 ft, the area would be 21 sq ft. If the room were 2 ft by 20 ft, the area would be 40 sq ft. If 3 ft by 19 ft, 57 sq ft. If 4 ft by 18 ft, 72 sq ft. Continuing in this fashion we find an area of 120 sq ft for a room **10 ft by 12 ft.**

16. One might use reasoning and table-making. For example: let one number be XY and the other YX and that XY + YX = 77 and XY – YX = 27. Assume that when the units are added there is no carry into the tens column.

X	Y	XY	YX	SUM	DIFFERENCE
0	7	7	70	77	–63
1	6	16	61	77	–45
2	5	25	52	77	–27
3	4	34	43	77	– 9
4	3	43	34	77	9
5	2	52	25	77	27

Thus we see that the two numbers are **52 and 25.**

17. One could reason that there are 4 choices for the chairperson and, for each one of these choices, there are 3 choices for secretary. So each of the 4 potential chairs could be matched with 3 potential secretaries. Thus there are 4 times 3, a total of **12** different officer combinations.

18. Responses will vary. The problem could be approached with the guess-check-revise strategy:
    a. Suppose you had 1 half dollar and 1 quarter. Then your remaining coins must be dimes, nickels, or pennies because another half dollar or quarter would give change for a dollar. Now, you could have up to 4 dimes and 4 pennies without change for a dollar. So the total is **$1.19.**
    b. Suppose you had a quarter. Then you could also have 4 dimes and 4 pennies and not be able to give change for a half dollar. So you could have **69 cents.** You could also have any number of half dollars.
    c. You could have 4 nickels and 4 pennies: **24 cents.** You could also have any number of dimes, quarters, or half dollars in any combination.
    d. You could have a nickel and 4 pennies, **9 cents** and any number of dimes, quarters, or half dollars.
    e. You could have 4 pennies, **4 cents**, or any number of nickels, dimes, quarters, or half dollars in any combination.

19. Responses will vary. One might reason the following way: At 4 AM the small hand is on the 4, the large hand on the 12. The minute hand will pass the hour hand as the clock goes to 5 AM, 6 AM, 7 AM, 8 AM, 9 AM, 10 AM, and 11 AM. The hands will be colinear at 12 noon, the passing point on the way to 1 PM. They will pass 1 final time on the way to 2 PM. Counting the passes, we see that the minute hand passes the hour and **9 times**.

20. Responses will vary. One might reason this way: The most efficient way to cut grass is to pass over each section of grass only once. So imagine that for each pattern the grass is cut into rectangular strips 30″ wide laid end to end. The cutter would have to walk the total length of the strips. But since the area of the lawn is the same for either pattern, the total length is also the same. Thus the distance walked is the same. Any difference would be in the mechanics of turning.

21. Applying the guess-check-revise strategy we find that row 1 = **1 2 3** and row 2 = **5 6 4** is a solution to the problem.

22. Responses will vary among students. Most frequently used would probably be guess-check-revise.

23. Responses will vary among students. Possibilities include: Will it be cheaper for Nathan to install the TV himself or to have the store install it? What is the cheapest TV Nathan can buy and have the cost of the TV, the bracket, and the installation be cheaper than the cost of the TV and bracket and installing it himself?

24. Responses will vary among groups of students.

25. #7 Suppose J is Jack's age. Then Chris is $J + 4$ years old. Together their ages total 30, so $J + (J + 4) = 30$ or $2J + 4 = 30$ or $2J = 26$, which means $J = 13$. So Jack is 13 years old, and Chris is 13 + 4 or 17 years old.

#8 As used in the solution of this problem, last refers to the terminal runner in a race. 'Second to last' is the same as 'next to last'. Now, one might draw a diagram and see that since Bella is fifth from last in a 21 person race, she is in 17^{th} place. Now, she must pass one person to be in 16^{th} place, a second person to be in 15^{th} place, and so on until she has passed 14 persons and is in third place.

#9 Write a few terms of the club membership and look for a sequence. The numbers joining are 1, 2, 3, 4, ..., 40. These numbers are in an arithmetic sequence. We notice that the 1^{st} and 40^{th} terms add to 41 as do the 2^{nd} and 39^{th}, as do the 3^{rd} and 38^{th}, and so on to the 20^{th} and 21^{st}. Thus there are 20 sums of 41, a total of **820 members.**

#10 There are 4 choices for the first letter in the in the arrangement and for each of these there are 3 choices for the second letter. For each of these 12 choices for the first two letters there are 2 choices for the third letter. Thus there are a total of **24 possible arrangements.**

#11 One might reason as follows: since Beth coaches a water sport and there is only one water sport, swimming, **Beth coaches swimming**. Because we know Anton's sister coaches soccer and because Diedre is the only remaining woman, **Diedre** must be Anton's sister as well as **the soccer coach.** We know **Anton** does not coach basketball so he must coach the remaining sport, **volleyball,** leaving **basketball to Cal.**

#12 Pam's initial balance was \$300. She then had a charge of \$5 and wrote checks for \$20, \$50, \$75, and \$15 or a total of \$160. So her balance was \$300 – \$5 – \$160 = \$135. After a deposit her balance was \$200, so \$135 + ______ = \$200. She deposited \$65.

26. Responses will vary among students.

27. Responses will vary among students. One might suggest to Angela that she is on the right track. She is getting closer to the answer, 24 to 9 is closer to 2 to 1 than is 21 to 6. She might increase her increment to 2 or 4 or 5 rather than the brute force of 1.

28. Responses will vary among students. One solution is: Let S = single deliveries, T = twin deliveries, R = triplet deliveries. Then S + T + R = 40 and T + R = 8. So S = 32 leaving 21 children born in multiple deliveries. So 2T + 3R = 21 and since T + R = 8, we have 2 (8 – R) + 3 R = 21. So R = 5 and T = 3. **There were 32 single babies, 3 sets of twins, and 5 collections of triplets.**

29. Pick any six balls and put three on each side of the balance. If this setup is in balance, use the balance to determine which of the two remaining balls is the heavier. If the original setup is not in balance take any two of the three balls on the heavier side and put them on the balance. If either is the heavier, it will move its side down. If the balance is level here, then the one of the 3 not on the balance scale is the heavier.

30. Response will vary among students. If B = slices eaten by Brenda, then 2B = slices eaten by Tina and 6B (3 times 2B) represents slices eaten by Gene.

31. Responses will vary among students. A possible solution is: To fence a rectangular plot one needs posts at the corners. Suppose the posts are placed four meters apart. So: how many posts are needed on each long side? How many total on the long sides? How many posts are needed on each short side? How many posts total on the short sides? How many posts total so far? Are any posts counted twice? How many? How can you adjust the total for these double countings? (6, 12, 5, 10, 22, Yes, 4 corner posts, subtract 4, answer: 18)

32. Responses will vary among pairs of students. One might reason a solution as follows. If the height reached on a particular day is greater or equal to 10' above the bottom, then the beetle will escape on that day. The starting position for each day is 1 ft higher than the day before. So, calling the bottom 10' and the top 0' make a table as:

Day	Start	Max	End
1	10	5	9
2	9	4	8
3	8	3	7
4	7	2	6
5	6	1	5
6	5	0	ESCAPE!!!

33. a. Consider any one of the five people. She will shake hands with 4 other persons. The total so far is 4 handshakes. Now consider a second person. There are 3 people remaining for her to shake hands with. The total is now 7 handshakes. Similar arguments give 2 handshakes for person 3, 1

handshake for person 4, and person 5 has already shaken hands with the other four guests. So the total number of handshakes is 4 + 3 + 2 + 1 = **10**. The solution to the second problem has the same logic. Simply replace persons with dots and handshakes with line segments.

b. Responses will vary among students.

34. Responses will vary among students. 'Orientation to the problem' is similar to 'Understand the problem', 'Produce relevant ideas and form and test hypotheses for solving the problem' is similar to 'Develop a plan'. The strategies are similar.

CHAPTER 1 REVIEW EXERCISES

1. Answers will vary among students.

2. The formulas are: **C2 = A2 × B2** and **C3 = A3 × B3**.

3. Responses will vary. Possibilities include:
   a. A counterexample is $10 \div 1 = 10$.
   b. A rectangle may have sides that are not equal, say 3 and 4. Thus it is not a square.
   c. Consider the product of 7 and 2. The product is 14, an even number.

4. a. The hypothesis is 'Morgan Freeman gives a good performance'. The conclusion is 'he will win the Oscar'.
   b. The hypothesis is 'the actress has won an Oscar'. The conclusion is 'she will be in high demand'.

5. The conclusion, obtained by **affirming the hypothesis,** is that angle C of figure ABCD is a right angle.

6. Conclusion: Nicole did not lay in the hot sun for 4 hours; **denying the conclusion.**

7. Responses will vary among students. Some examples are:
   a. My aunt lives in Austin. Therefore she lives in Texas.
   b. I don't know anyone who lives in Texas. Thus I know no Austinians.
   c. Since Joe lives in Texas, Joe lives in Austin.
   d. Since I do not live in Austin, I don't live in Texas.

8. Recall that $p \rightarrow q$ is false only when p is true and q is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

   a. I live in El Paso Texas.
   b. I spent my money on concert tickets and did go to the game.

9. The conjunction p and $\sim p$ is false because a conjunction requires that both parts be true. However, p and $\sim p$ cannot be true simultaneously.

10. The disjunction p or $\sim p$ is true. If p is false, then $\sim p$ is true. For the disjunction to be true, only one part of it must be true.

11. Converse: If you live in LA, you live in California. (True)
Inverse: If you do not live in California, you do not live in LA. (True)
Contrapositive: If you do not live in LA, you do not live in California. (False)

12. Two chances: (1) Work overtime on Sunday (2) Work on Tuesday night and on Wednesday night

13. a and b. i. 22, 33, 44, 55, 66, 77, 88, 99, **110, 121, 132: arithmetic, d = 11.**
ii. 1, 11, 12, 22, 23, 33, 34, 44, **45, 55, 56: neither**, sequence is formed by alternately adding 10 and 1
iii.2, 6, 18, 54, 162, 486, 1458, **4374, 13,122, 39,366: geometric, r = 3.**

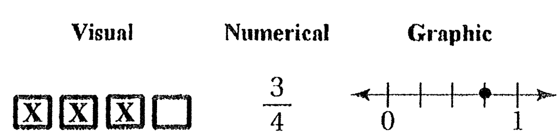
14. a. The next statement is $\mathbf{7 \div 0.5 = 14}$.
b. The next statement is $\mathbf{5 \times 64 = 320}$.

15. a. The first column is an arithmetic sequence with $d = 1$. The second column is an arithmetic sequence with $d = 2$. Continuing these sequences we get the next three rows as: **7 13; 8 15; and 9 17.**
b. We also note that the entries in the second column are 1 less than twice the corresponding first column entries. So the 25^{th} row would be **25 49.**

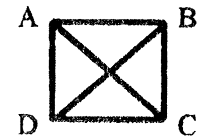
16. Responses will vary. Among the problem solving strategies are: use a spreadsheet, use GES, use a calculator for guess-check-revise, look for patterns, apply deduction, apply proportional reasoning, and make a model.

17. Responses will vary. A possibility is:
a. Use a guess-check-revise strategy.
b. Parking–\$10, concert ticket–\$40, total \$50 is low. So revise the cost of the parking up. Parking–\$15, concert ticket–\$60, total \$75 is still low. So revise the cost of the parking up. Parking–\$18, concert ticket–\$72, total is \$90. So parking cost **\$18, and the concert ticket cost \$72**.

18. Responses will vary among students. A potential response might be: Verbal - suppose I have four boxes on a form and I select three. Then three fourths of my boxes have been selected.

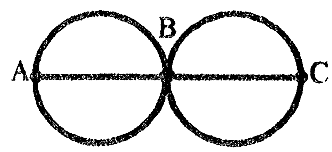


19. These two problems are essentially the same because they can be represented by a common geometric model. The vertexes in the model can represent either the people or the desks, the segments can represent the cables or a pairing of riders.



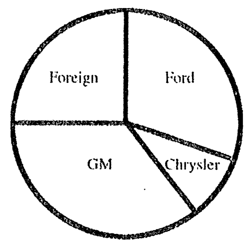
20. Responses will vary. Possibilities include: 0.5, XXXXXXXX.

21. The points A, B, and C represent the cities. The arcs and diameters represent the roads.



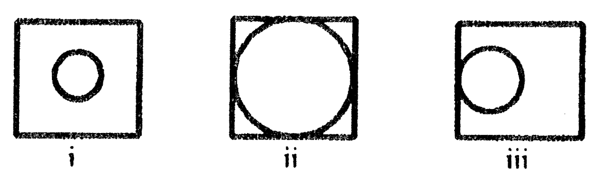
22. Responses will vary among students.

23. A possibility is the circle graph:



24. a. The (areas, perimeters) are: for 3×8: (**24, 22**); for 2×11: (**22, 26**); for 1×14: (**14, 30**).
    b. The areas decrease.
    c. The perimeters increase.
    d. Responses will vary. One possibility is: As the areas decrease the perimeters increase.

25. a. The common aspect of the figures is that the circle must be inside the square. The orientations of the square with respect to the circle may differ.
    b. Responses will vary. A set of possibilities is:



26. The cheeseburgers cost \$3 + \$2 + \$1.50 + \$1.50 = \$8. The drinks cost an additional \$4, bringing the total to \$12. The tax is 0.07 x \$12 = \$0.84. Thus the total bill is \$12.84. Change from \$20.00 would be \$20.00 - \$12.84 = \$7.16.

27. Fill in the following table:

Number of Items	Cost Up Front Plan	Straight \$40 Cost
0	100	0
1	120	40
2	140	80
3	160	120
4	180	160
5	200	200

The breakeven point is **5 items purchased.**

28. Answers will vary among students. Note that the conditionals in the ad have no conclusions regarding smooth skin. A possibility is: First of all, the reader misinterpreted the word "want" in the ad as "will" and accepted the statement: "If you want to keep your skin smooth, then use Wrinkleflee" as being equivalent to "If you use Wrinkleflee, then your skin *will* stay smooth." The reader has assumed the converse. This is a logical error. The second implication of the ad is: 'if you use Wrinkleflee, then you treat your skin with loving care'. The reader assumed the inverse of this statement. This is another logical error.

29. Answers will vary among groups. Problem-solving skills refer to the ability of individuals or groups of persons to implement the problem-solving process with the end result of arriving at suitable strategies to solve particular problems.

30. Responses will vary among students. Briefly, **inductive reasoning** generalizes from a pattern. It rests upon **experimentation.** When a common characteristic is identified in some specific cases, a generalization is made that that characteristic belongs to all members of the class containing the cases investigated. **Deductive reasoning** is based upon **argument**, not experimentation. Certain general statements are agreed to be true and then specific cases are shown to meet the requirements of the general statements. If this can be shown then the general conclusion is applied to the specific instance.

31. Caitlin is correct. She could pass the same person more than once and someone she passed could repass her to finish ahead of her.

32.

Day	Payment that day	Total payment to date
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	128	255
9	256	511
10	512	1023
11	1024	2047
12	2048	4095
13	4096	8191
14	8192	16383

a. The table above illustrates column extension patterns in all three columns. The first column is the number of days worked, and it increases by 1 each time. The second column shows how much the diver is paid each day, and the amount doubles every day. The third column shows the total amount of money the diver has earned. Each new amount is obtained by taking the prior day's total and adding the new money earned that day (second column.) This table illustrates row relationship patterns in two ways. Looking across each row, we see that the payment for the day is obtained by finding 2^{n-1}, where n is the day. Moving across to the third column, the value is obtained by doubling the value in the second column and subtracting 1.

b. Responses will vary. In general, column expansion can be easier to locate and extend patterns. However, column extension leads to recursive formulations in which you must know the preceding value to determine the next value. You cannot jump form the tenth entry to the fiftieth entry using column extension. Row relationships can be more difficult to determine, but they lead to formulations that will give the value for any row.

SECTION 2.1

1. A one-to-one correspondence can be established between two sets if they have the same number of elements. Sets **B and E** both have 4 elements. Both sets **A and D** have zero elements. They are different representations of the null set.

2. Equivalent sets have the same number of elements; equal sets have exactly the same elements although not necessarily in the same order.
   a. Sets **A and B, C and B, D and B** are neither equal nor equivalent.
   b. Since equal sets must be equivalent, there is no solution.
   c. Sets **A and D** and sets **C and D** are equivalent but not equal.
   d. Sets **A and C** are equivalent and equal.

3. n(S) represents the number of elements in a set.
   a. There is **1** element in the set {yellow].
   b. There are **0** elements in the set ∅, the null set.
   c. There are an infinite number of elements in the set {3, 6, 9, 12, ...}.
   d. There are **5** elements in the set {1, 2, 3, 4, 5}.
   e. There is **1** element in the set {∅}.

4. Suppose S1 and S2 are sets and $x = n(S1)$ and $y = n(S2)$. If S1 and S2 can be placed in a one-to-one correspondence, then $x = y$. If, when the elements of S1 and S2 are paired, S1 has unpaired elements, then $x > y$. If S2 has unpaired elements then $x < y$.
   Let S1 = {a, b, c, d, e, f, g, h}. n(S1) = 8. Let S2 = {m, n, o, p, q, r}. n(S2) = 6. Pairing a-m, b-n, c-o, d-p, e-q, f-r, we see that S1 has elements g and h remaining. Thus 8 > 6.

5. A set S1 is a subset of a set S2 whenever every element of S1 is also an element of S2. A set S1 is a proper subset of a set S2 whenever every element of S1 is also an element of S2 and S2 also contains elements not in S1. Every set is a subset of itself, but not a proper subset. The null set is a proper subset of all sets except itself. It is a subset of itself. Applying these concepts we see that:
   a. **A** is not a proper subset and not a subset **C; B** is not a proper subset and not a subset **A; B** is not a proper subset and not a subset **C**.
   b. No sets meet this conditions. A proper subset is a subset.
   c. A, B, and C are all subsets of themselves. So pairs **A-A, B-B, C-C** meet the conditions.
   d. Any proper subset is also a subset. So **A-B, C-A, C-B** meet the conditions. A, B, C, are all proper subsets of the universal set U.

6. If $x = n(s)$, then the number of subsets of S is 2^x. The subsets of S = {1, 2, 3, 4} are **{1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 3, 4}, {2, 3, 4}, {1, 2, 4}, {1, 2, 3, 4}, {}.**

7. Answers will vary. S ⊆ T means that S is either a proper subset of T or else equal to T. S ~ T means that S and T have the same number of elements. Together these imply that S = T. R ⊂ S means that R is a proper subset of S: all elements of R are in S but not all elements of S are in R. So a solution is: R = {a,b}, S = {a,b,c}, T = {a,b,c}.

8. $P \subseteq Q$ means that every element of P is also in Q. $Q \subseteq P$ means that every element of Q is also in P. Together these imply that P and Q have exactly the same elements because if P has elements not in Q, $P \subseteq Q$ is false and if Q has elements not in P, then $Q \subseteq P$ is false.

9. a. The set Z = {z} is a subset of the set S = {w,x,y,z} because all of the elements of Z are contained in S.
   b. The set A = {a} is not a subset of S because there is at least one element of A that is not in S.

10. a. One may argue that the set N = { } is a subset of S = {w,x,y,z} as follows: If there are no elements of N that are not also in S, then N is a subset of S. But there are no elements in N. Thus there cannot be elements in N not in S. Thus N is a subset of S. One might also argue indirectly: either N is or is not a subset of S. If N is not a subset of S then there must be some element in N that is not in S. But there are no elements in N. Thus there are no elements in N not in S. Thus N cannot be not a subset of S. Thus the remaining possibility must be true, that is N IS a subset of S.
    b. The arguments above hold for any set S including S = { }.

11. a. S = {w,x,y,z} is a subset of itself because there is no element of S that is not an element of S.
    b. The argument may be applied to any set S.

12. a. The complement of A, $\overline{A}$, is the set of all elements in U that are not in A. Thus, $\overline{A} = \{a, b, c, d, e\}$.
    b. $\overline{B} = \{a, b, c, d, e, f, g\} = U$
    c. $\overline{C} = \{b, d, f, g\}$
    d. $\overline{U} = \emptyset = B$

13. a. Cardinal–tells how many
    b. Nominal–tells which one; Ordinal–tells what place
    c. Nominal–tells which one
    d. Ordinal–tells what place

14. The proper subsets of S = {a,b,c} are: **{ }**, **{a}**, **{b}**,**{c}**, **{a,b}**, **{a,c}**, **{b,c}**. The whole numbers associated with each of the proper subsets of S are less than the whole number associated with S: 0, 1, and 2 are less than 3.

15. a. S = {0, 1, 2, 3, 4, 5}. This is a finite subset of the whole numbers and it does have a greatest element, 5.
    b. T = {6, 7, 8, 9, ...}. This subset of the whole numbers does not have a greatest element.
    c. There is no infinite subset of the whole numbers that has a greatest element.

16. Every subset of the whole numbers does have a least member.

17. The subsets of {red, green, blue} are:**{ }**, **{red}**, **{green}**, **{blue}**, **{red, green}**, **{red, blue}**, **{green, blue}**, **{red, green, blue}.**

18. The counting numbers are a proper subset of the whole numbers. The whole numbers are the union of the counting numbers and the set {0}.

19. All sets that are in a one-to-one correspondence are defined as equivalent sets and are associated with the same whole number.

20. The ideas of subset and proper subset are used to illustrate the relations 'greater than' and 'less than' between whole numbers. If some set is equivalent to a proper subset of some other set, then the number associated with the first set is less than the number associated with the second set.

21. The sets A = {a,b,c,d}, B = {w,x,y,z}, and C = {a,b,y,z} are all equivalent to the set S = {a,b,c,d}.
    a. Sets C and S show that equivalent sets do not necessarily have all elements in common.
    b. Sets B and S show that equivalent sets do not necessarily have any elements in common.
    c. Equivalent sets must have the same number of elements.

22. The choices of toppings form a set with four elements. Suppose the elements are a, b, c, and d. The subsets of the set of toppings are { }, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b,c,d}. The choices of toppings, including no topping, are all the subsets of the topping set.

23. a. The empty set is the only subset of itself.
    b. The empty set has no proper subsets because it is the only subset of itself;
    c. One can conclude that there is no whole number less than the number associated with the empty set, the number 0.

24. a. Equal sets are necessarily equivalent because if two sets contain exactly the same elements then they contain the same number of elements.
    b. Equivalent sets are not necessarily equal because just because two sets have the same NUMBER of elements, the elements are not necessarily identical.
    c. Equal sets contain exactly the same elements and therefore the same number of elements. Equivalent sets contain the same number of elements, but not necessarily the same elements.

25. a. **False**: equivalent sets are not necessarily equal. Consider A = {a,b} and B = {c,d}.
    b. **True**: if two sets do not contain the same number of elements they cannot contain exactly the same elements.
    c. **True**: if two sets contain exactly the same elements those sets contain the same numbers of elements
    d. **False**. Consider A = {a,b} and B = {c,d}.

26. a. **False**. Consider A = {a,b} and B = {c,d,e}.
    b. **False**. Consider A = {a,b} and B = {c,d,e}.
    c. **True**: since A is a proper subset of B, then all the elements of A are in B and B contains elements not in A. Thus there are more elements in B than in A.
    d. **True**: If A is a subset of B then either A and B contain exactly the same elements, in which case the cardinal numbers are equal or all of the elements of A are also in B and B has some elements not in A. In this case the cardinal number of A is less than that of B.

27. The librarian means that faculty may check out only some, not all, of the newly acquired books.

28. a. There is **one** subset of P ={ }. That subset is P itself.
 b. There are **2** subsets of R = {a}: { } and {a}.
 c. There are **4** subsets of S = {a,b}: { }, {a}, {b}, {a,b}.
 d. We see in part a that when a set has no elements there is $1 = 2^0$ subsets; in part b that a set with 1 element has $2 = 2^1$ subsets; in part c that a set with 2 elements has $4 = 2^2$ subsets. We generalize that a set with n elements has 2^n subsets.

29. The elements of the sets W and N may be paired: 0-1, 1-2, 2-3, 3-4, 4-5, ..., $(n-1)$-n, n-$(n+1)$, ...

30. Responses will vary. One possibility is: Let A and B be sets and suppose $a = n(A)$ and $b = n(B)$. Now, either A and B are equivalent or they are not. If they are equivalent, then $n(A) = n(B)$ and $a = b$. If they are not equivalent, then when we attempt to pair elements of A with elements of B we will find elements remaining either in A or in B. If elements remain in A we have $n(A) > n(B)$ and $a > b$. If elements remain in B we have $n(A) < n(B)$ and $a < b$.

31. Let the persons available to serve on the committee comprise the set S = {A, B, C, D, E}. A subcommittee must contain at least two persons and may be a committee of the whole. So the number of possible subcommittees is the number of subsets of S excluding the empty set and the 5 singles: $2^5 - 1 - 5 = 32 - 6 = 26$.

32. Responses will vary. They may include discrete situations described with whole numbers such as students in a class, cards in a hand, persons invited to a party. Continuous situations might include the metric measures of length, mass, time, force.

33. Responses will vary. Possibilities include: sets of students and sets of desks, persons invited to a party and party favors, plates and persons at the dinner table. Some problems are trivial such as more Christmas cards than stamps; others more serious as more children than doses of vaccine.

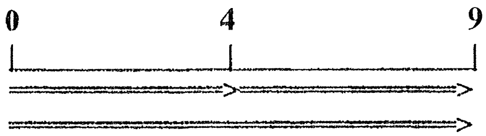
SECTION 2.2

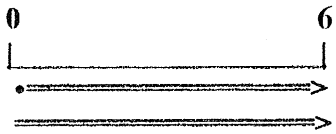
1. a. The union is E = {those people who are either more than 20 years old or those persons enrolled in college (or both)}.
 b. The union is C = {2, 4, 6, 8}
 c. The union is G = {1, 3, 5, 7, 9, ... , 37, 39, 41, 42, 43, 44, ...}.

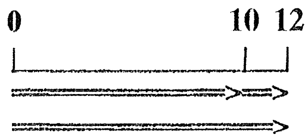
2. a. The intersection is E = {those persons enrolled in college who are more than 20 years old}.
 b. The intersection is C = {4}.
 c. The intersection is G = { }.

3. a. The single element common to both sets is e. So $R \cap S$ = **{s, t, p}**.
 b. The set consisting of the elements in both R and T is **{s, t, o, p, l, i, e, n}**.
 c. The set consisting of the elements in both S and T is **{p, l, i, s, t, e, n}**.
 d. $(R \cap S) \cap T$ = {s, t}.
 e. $R \cap (S \cap T)$ = {s, t}

4. a. {@,@,@,@} ∪ {$,$,$,$,$} = {@,@,@,@,$,$,$,$,$}.
   b. { } ∪ {&,&,&,&,&,&} = {&,&,&,&,&,&}.
   c. Show the union between 2 disjoint sets, one with 10 elements and the other with 2 elements.

5. a. 

   b. 

   c. 

6. If 2000 tickets to a concert were pre-sold and another 356 were sold at the door, how many tickets total were sold?

7. If 2000 miles were traveled on the interstate and 356 miles on secondary roads, what total distance was covered?

8. **No,** because the set of those students bringing apples and the set of those students with peanut butter sandwiches may not be disjoint; some students may have brought both.

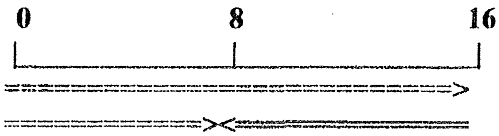
9. Examples will vary.
   a. This is an application of the **commutative** property of addition: 2 + 8 = 8 + 2.
   b. This is an application of the **identity** property of addition: a + 0 = 0 + a = a.
   c. This is an application of the **associative** property of addition:
      8 + 7 = 1 + 7 + 7 = 1 + (7 + 7) = 1 + 14 = 15.
   d. This is an application of the **uniqueness** property of sums.

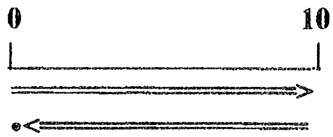
10. a. 14 < 20 because there is a unique counting number, 6, such that 14 + 6 = 20.
    b. 3000 > 2532 because there is a unique counting number, 468, such that 2532 + 468 = 3000.
    c. There is a counting number, 6, such that 0 + 6 = 6.

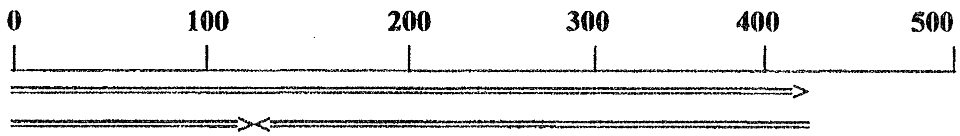
11. a. {♦, ♦, ♦, ♦, ~~♦, ♦, ♦, ♦, ♦~~}
    b. {~~♣, ♣, ♣, ♣, ♣, ♣~~}
    c. $\{1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, \cancel{8}, \cancel{9}, \cancel{10}, \cancel{11}, \cancel{12}\}$.

12. a. commutative property of addition
    b. commutative property of addition
    c. associative property of addition

13. a. $3 + 1 = 4$ $\quad$ $4 - 1 = 3$
$1 + 3 = 4$ $\quad$ $4 - 3 = 1$
b. $15 + 10 = 25$ $\quad$ $25 - 10 = 15$
$10 + 15 = 25$ $\quad$ $25 - 15 = 10$

14. a.


b.


c.


15. The definition of subtraction is: $a - b = c$ if and only if there exists a unique whole number, c, such that $a = b + c$. (a) $18 = n + 8$ (b) $25 = 15 + x$ (c) $y = 129 + 83$ (d) $a^2 = 30 + a$

16. a. $8 = n - 7$ and to $7 = n - 8$.
b. $14 = 25 - x$ and to $x = 25 - 14$.
c. $r = t - s$ and to $s = t - r$.

17. A student had 12 lottery tickets. Five of these were known losers. How many tickets remained?

18. To complete a project, Sue cuts a 5-foot board from a 12-foot board. How long is the piece of board that she has left over?

19. A professor gives a quiz with 12 questions. If 5 of the questions are essay questions, how many of the questions are not essay questions?

20. Lashonda is 12 miles from school. Ross is 5 miles from school. How much farther does Lashonda have to travel to school than Ross?

21. The professor has written 5 essay questions for a quiz. How many more questions does the professor need to write so that the quiz will have 12 questions?

22. Responses to this question will vary among students. The results of the computations are: 200, 200, 200, 200, 200, 200, 200, 200, 200 and 20, 30, 40, 50, 60, 70, 80, 90, 100.
   a. For the addition sequence suppose the first sum, S, is given by $x + y$. If pairs of addends are $(x + d)$ and $(x - d)$ then S will also be the sum. For the difference sequence, if the first difference, d, is given by $x - y$, then differences of the form $d + 10n$, n a natural number is produced by subtracting $y + n$ from $x + 11n$.
   b. These patterns can be used to illustrate place value, associativity, commutativity. For example, consider the subtraction pattern: $d + 10n = (x + 11n) - (y + n)$. Applying place value we see that $x + 11n = x + 10n + n$. So $(x + 11n) - (y + n) = x + 10n + n - y - n = x - y + 10n$.

23. a. The key sequence would be: + **1** = = + **10** = = = = + **100** = = =.
   c. The display would be: 1, 2, 12, 22, 32, 42, 142, 242, 342.
   d. The properties of **commutativity and associativity** are applied.

24. a. The **set is closed under addition** because the sum of two multiples of 3 is a multiple of 3. A multiple of 3 may be defined as 3 times some natural number. So $E_1 = 3n_1$ and $E_2 = 3n_2$. Thus $E_1 + E_2 = 3(n_1 + n_2)$, a multiple of 3.
   b. Zero is the additive identity element because for any whole number, w, $w + 0 = w$.
   c. Since addition is commutative within the set of whole numbers and multiples of 3 are whole numbers, addition is commutative within the set of multiples of 3.
   d. Addition is associative by the argument of part c.

25. a. The set of odd whole numbers is not closed under addition. For example, $3 + 5 = 8$.
   b. The natural numbers are a subset of the whole numbers that lacks an additive identity element.
   c, d. These subsets of the whole numbers do not exist.

26. Symbolically, the student is saying: $(20 - 10) - 5 = (20 - 5) - 10$. This is true. However, commutativity would be expressed: $20 - (10 - 5) = 20 - (5 - 10)$. But this is not correct. $20 - (10 - 5) = 15$ and $20 - (5 - 10) = 25$.

27. a. The properties used are associativity and base-ten place value: $9 + 6 = 9 + (1 + 5)$ $(9 + 1) + 5 = 10 + 5$. But in base-ten, 10 is represented by a '1' in the second position. Thus $10 + 5$ is represented as 15.
   b. The response to this question depends upon the insights of the students. A possibility is: the sums of $1000 + 500$, $100 + 50$, and $10 + 5$ have the same non-zero digits.

28. a. A set, S, is closed under an operation if the result of the operation on any element(s) of S is also in S. The table, together with knowledge of place value, can be used to obtain the sum of any 2 whole numbers. Since the sum is a whole number, the set of whole numbers is closed under addition.
   b. As in a, we can use the table and place value to show that 0 added to any whole number results in the number. Thus 0 is an identity element.
   c. Since the top right of the table is the mirror image of the bottom left, $a + b = b + a$.
   d. This response depends on the insights of the students. They might suggest that because of commutativity only half of the addition facts need be memorized.

29. $a < b$ if and only if a is to the left of b on the number line.

30. The responses to this problem will vary among students. Representative responses might be:
    a. Addition: How many students had lunch at school on Monday? Subtraction: How many more students had lunches from home on Monday than on Tuesday?
    b. $11 + 12 = ?$ $11 - 6 = ?$ Subtraction problems can also be written as $11 = 6 + ?$

31. Responses will vary among students. Students might point out that subtraction problems can be rewritten as missing addend addition relations. Separating a set into two sets is modeled by physically removing objects from some set. The objects removed may be placed into a one-to-one correspondence with some third set. Thus 2 sets are formed: one from the objects removed and the other by the objects that remain. When comparing sets one may remove objects from the two sets in pairs, one from each set, to determine which, if either, has elements remaining after one of the sets has been reduced to the null set.

32. The application of the mathematical relation $14 - 8 = 6$ to a variety of physical situations demonstrates the generality of mathematics. This generality is one of the characteristics that make mathematics so useful.

33. a. If a increases or c decreases, then e increases. When a decreases or c increases, then e decreases. If a increases and c decreases, e increases.
    b. Tests applied will vary among students.
    c. See part a.
    d. e decreases if: a decreases, c increases, a increases by less than c increases, a decreases by more than c decreases. e increases if: a increases, c decreases, a increases by more than c increases, a decreases by less than c decreases.
    e. Responses depend on interactions among students.

34. a. This set is **not closed:** $1 + 1 = 2$, 2 is not in the set.
    b. This set is **closed** under addition. $0+0=0$.
    c. This set is **closed** under addition: the sum of any 2 numbers ≥ 10 is > 10.
    d. This set is **not closed** under addition: $2 + 6 = 8$. 8 is not a member of the set.
    e. This set is **closed** under addition: $S = 100x + 110y = 100(x + y)$, a multiple of 100.

35. **False.** The set of odd numbers is an infinite set not closed under addition because $1+3=4$.

36. a. The system **is closed** under *. *ing of any 2 elements in the set $\{a, b, c, d\}$ results in an element of the set.
    b. There is **no identity property** of this set. The identity element must be unique. However, $d * a = d, d * b = d$.
    c. The operation * is **not commutative:** $c * a = a$ but $a * c = c$.
    d. The operation * is **not associative:** $(b * b) * b = c * b = b$ but $b * (b * b) = b * c = a$.

37. Responses may vary. One possibility is:

#	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

This system is analogous to addition mod 4.

38. Responses depend upon prior experiences of students.

39. a. If the condition $k > 0$ were omitted but we still stay in the set of whole numbers, we have the situation that although a equals a, $a + 0 = a$ which implies that a is greater than a.
    b. The definition of 'less than' might be stated: a is less than b if and only if there exists some whole number $k > 0$ such that $a = b - k$.

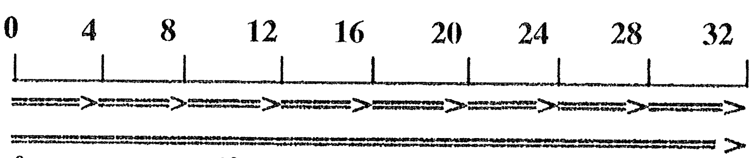
40. The response to this question depends on interactions among students. Students might rest their argument upon a physical operation different from addition: removing objects from a set or moving left on a standard number line.

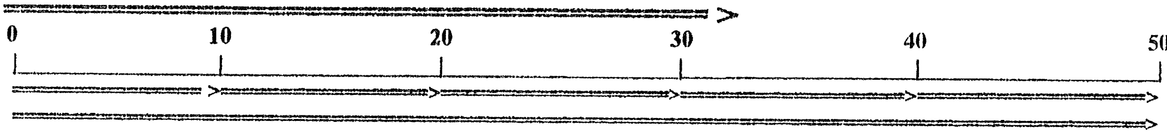
41. Responses will vary. Possible ideas are: Scribes might have treated subtraction as missing addend problems. Suppose the scribe wanted to determine the difference 14 – 6. Interpreting the problem as 14 = 6 + ? the scribe would locate that 14 in the body of the table that had one addend of 6 and find the corresponding other addend.

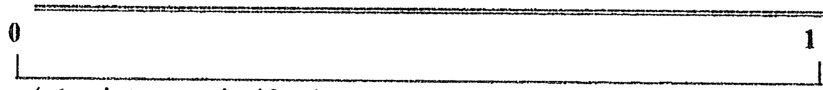
42. Responses will vary. The numbers required are the negative integers. Each of these is related to a natural number such that –n + n = 0. Negative integers model deficits, losses, temperatures below zero, elevations below sea-level, and a myriad of other situations.

SECTION 2.3

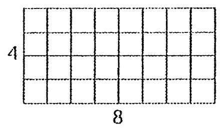
1. To verify the operations one would:
   a. show the union of 8 disjoint sets each with four elements.
   b. show the union of 5 disjoint sets each with 10 elements.
   c. show the union of 6 empty sets.

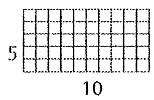
2. a.


b.


c.


• (6 points are coincident)

3. a. 

   b. 

   c. The is no array of one dimension equal to 0.

4. If each of 4 children received 30 pieces of candy on Halloween and put all the candy in a bag to share with their classmates, how many pieces of candy were in the bag?

5. If 4 students are running a relay in which each student runs 30 meters, how long is the race?

6. a. A = 12 units • 6 units = **72 square units.**
   b. 42 units • 1 unit = **42 square units.**
   c. $\frac{h}{2}$ units • $4b$ units = $\left(\frac{h}{2}\times 4b\right) = 2hb$ **square units.**

7. a. 4 chairs • 6 fabrics = **24** different chairs

8. 3 chocolates • 5 cheeses • 3 coffees = **30** baskets

9. w wrapping paper types × c bow colors = 64 paper-bow combinations

10. Answers will vary among students. Representative responses might be:
    a. Students may chose to do community service in one of 25 cities at one of 4 charitable organizations with facilities in each of the cities. How many choices does each student have?
    b. If a student arranged her collection of campaign buttons in a pattern of 25 rows and 4 columns, how many buttons did she have?
    c. What is the area of a strip garden 25 feet by 4 feet?

11. a. Multiplication is commutative: $a \times b = b \times a$.
    b. There is a **zero property** of multiplication: for each whole number a, $a \times 0 = 0 \times a = 0$.
    c. In the whole numbers multiplication is distributive over addition. Suppose that $c + 1 = b$. Then $a \times b = a \times (c + 1) = ac + a$.
    d. The whole numbers are **closed** under multiplication: the product of 2 whole numbers is a unique whole number.
    e. Associative property of multiplication

12. a. $(20+5)\times 3 = (20\times 3)+(5\times 3) = 60+15 = \mathbf{75}$.
    b. $4\times(5+6) = (4\times 5)+(4\times 6) = 20+24 = \mathbf{44}$.
    c. $(x+10)(3x+2) = (x)(3x)+(x)(2)+(10)(3x)+(10)(2) = 3x^2+2x+30x+20 = 3x^2+32x+20$.

13. A teacher has a bank of 24 test questions and wishes to create tests with 6 questions. How many tests are possible?

14. A shop teacher cut a 24 inch piece of wire into 6 inch segments. How many pieces were cut?

15. A teacher has a bank of 24 test questions and wishes make 6 tests with the same number of questions. How many questions will be on each test?

16. A shop teacher had a piece of wire 24 inches long. He bent the wire into 6 pieces of equal length. How long was each straight piece?

17. A teacher made 6 tests with the same number of questions. The total number of questions was 24. How many questions were on each test?

18. This question requires calculator activity by the students.
   a. $143 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 - 11 = 0$. So $143 \div 11 = \mathbf{13}$.
   b. $108 - 9 - 9 - 9 - 9 - 9 - 9 - 9 - 9 - 9 - 9 - 9 - 9 = 0$. So $108 \div 9 = \mathbf{12}$.
   c. Similarly, if 52 is subtracted from 1404 twenty-seven times, the result is 0. So $1404 \div 52 = \mathbf{27}$.

19. a. $18 = 6 \times n$; $n = \mathbf{3}$.
   b. $25 = 5 \times X$; $X = \mathbf{5}$.
   c. $y = 42 \times 126$; $y = \mathbf{5292}$.
   d. $0 = b \times c$. $\boldsymbol{c} = \mathbf{0}$ (b cannot be 0 because it is the denominator of the original expression. Thus b is any natural number.)

20. a. $15 = n \div 3$; $3 = n \div 15$.
   b. $9 = 9 \div y$; $y = 9 \div 9$.
   c. $r = t \div s$; $s = t \div r$.
   d. $0 = 0 \div 8$.

21. a. $26 \div 3 = \mathbf{8}\ \boldsymbol{r}\ \mathbf{2}$ because $8(3) + 2 = 26$.
   b. $292 \div 21 = \mathbf{13}\ \boldsymbol{r}\ \mathbf{19}$ because $21(13) + 19 = 292$.
   c. $4 \div 7 = \mathbf{0}\ \boldsymbol{r}\ \mathbf{4}$ because $0(7) + 4 = 4$.

22. a. i The identity symbol for addition is the zero symbol for multiplication.
      ii. Multiplication and addition are associative, commutative, closure, and have an identity property.
      iii. Multiplication has a distributive and zero property, addition does not.
   b. Neither subtraction nor division is closed, associative, commutative, has an identity element or a zero element. Division does distribute over both addition and subtraction.

23. Predictions will vary among students.

	predicted n(C)	C	actual n(C)
a.		{ } (there are no elements in { } to pair with elements of the second set. Thus there are no ordered pairs and C is empty.)	**0**
b.		{(r, a), (r, b), (s, a), (s, b), (t, a), (t, b)}	**6**
c.		{(a, 5), (a, 6), (a, 7), (a, 8), (a, 9)}	**5**

d. $4 \times 0 = 0$; $3 \times 2 = 6$; $1 \times 5 = 5$.

24. It is not necessary that the sets be disjoint. Let S = {a,b}, T = {a,b}. S × T = {(a,a),(a,b),(b,a),(b,b)}.

25. To determine the value of a particular place within the numeral the appropriate power of 10 is multiplied by the face value of that place.

26. a. Essentially two properties of multiplication are applied: the uniqueness of a product and the distributive property of multiplication over addition.
    b. $12 \times 64 = (10 + 2) \times 64 = (10 \times 64) + (2 \times 64) = 640 + 128 = 768$.

27. a. $\mathbf{4r(5 + 4)}$.
    b. $\mathbf{9(3a^2 + 9 + 1)}$.
    c. $\mathbf{2(12a + 7b + 10c)}$.

28. Responses will vary among students. Representative answers are:
    a. Use of the table for basic multiplication facts together with knowledge of place value shows that the product of any 2 whole numbers is a whole number. Thus the set of whole numbers is closed under multiplication.
    b. As in *a*, for every whole number, w, $1 \times w = w \times 1 = w$.
    c. As in *a* and *b*, for any two whole numbers a and b, we see that $a \times b = b \times a$.
    d. There is an element, 0, such that for any whole number e, $0 \times e = e \times 0 = 0$.
    e. Consider the product 7×6, given in the table as 42. Now $7 \times 6 = 7 \times (4 + 2) = 7(4) + 7(2)$ by the distributive property. It is seen in the table that $7(4) = 28$ and $7(2) = 14$. By addition $28 + 14 = 42$.
    f. This response depends upon the insights of the students.

29. Responses will vary among students. They might be based upon the concept that division problems can be rewritten as 'missing factor' problems. The table can be used to find the missing factor.

30. Responses will vary among students. Representative answers are:
    a. In separating a set into a known number of sets which are to be equivalent, one can think of a process of removing the elements from the original set one by one and placing an element into each of the numbered sets until the original set is empty. When fixing the number of objects to be placed in the factor sets, one can think of removing elements from the original set in groups of the given number. The first situation is related to question 15, the second is related to question 13.
    b. The situations are similar in that they are symbolically represented: $a \div b = c$. See question 13–17.

31. a. If the values of q and r that satisfy $(q \times 12) + r = 64$ are restricted to the whole numbers, the (q, r) pairs are: (0, 64), (1, 52), (2, 40), (3, 28), (4, 16), (5, 4).
    b. Only the pair $q = 5$, $r = 4$ satisfies the division algorithm because when dividing 64 by 12 the remainder must be less than the divisor.
    c. If the restriction were dropped any of the above (q, r) pairs would satisfy the division algorithm and the quotient would not be unique.

32. Responses will vary. Example are:
    a. Four sets of numbers of the form $(a + b) \div c$ are: $(4 + 6) \div 2$; $(10 + 15) \div 5$; $(6 + 9) \div 3$; $(14 + 49) \div 7$.
    b. a and b must be multiples of c because of the restriction of whole number quotients for each division expression.
    c. The property is valid in the format described in b because of the whole number quotient restriction. If this restriction is dropped, a more general property can be considered as illustrated by: $17 \div 5 = (6 \div 5) + (11 \div 5) = (1 \text{ r } 1) + (2 \text{ r } 1) = 3 \text{ r } 2$.

33. The responses to this questions will vary among students and will depend on the insights developed by the students. Students may notice that the table is symmetric with respect to the upper-left to lower-right diagonal because of the commutative property.

34. Responses will vary among students. Representative answers are:
    a. How many students responded that their favorite activity was reading? How many students responded that either crafts or sports are their favorite leisure activity? What is the number of symbols needed to represent those students who have audio-visual recreation as their favorite activity? If boxes are used to represent the students declaring audio-visual recreation as their favorite, how many students would not be represented?
    b. $R = 7 \times 5$. $CS = (4 + 3) \times 5$. $33 \div 5 = q$ remainder r, find q. Different equations could be obtained for the first two by applying the commutative property or rewriting the equations in division form: for example, $R \div 7 = 5$.

35. **Yes, it could be true.** When there were 4 candy dishes the possible remainders were 0, 1, 2, 3. If these occurred with equal frequency, then Eleanor ate an average of one and a half pieces of candy for each bag. But with 7 dishes, the possible remainders are 0, 1, 2, 3, 4, 5, 6 which gives an average of 3 candies left after filling the dishes.

36. Responses will vary among students. Representative responses are:
    a.

Increase	decrease	constant	effect on q	effect on r (potential)
a, b, or a and b		c	increase	constant
c		a, b	decrease	increase
a, b, or a and b	c		increase	
c	a, b		decrease	

    increase/decrease combinations depend on the relative amounts of increase and decrease.
    b. This part requires student activities.
    c. If a or b or both increase but c is constant, then q will increase and the largest possible remainder is constant. But if a or b or both decrease and c remains constant, then q decreases but the largest possible remainder is constant. If a and b are constant but c increases, then q decreases but the largest possible remainder increases. On the other hand, if a and b are constant but c decreases, then q increases but the largest possible remainder decreases.
    d. q decreases if a or b or both decrease or if c increases. q increases if a or b or both increase or if c decreases.
    e. This response depends upon student interactions.

37. **All these sets except (a) are closed under multiplication**. First consider the set of even whole numbers. 0 times any element is 0 and therefore is in the set. A whole number is even if and only if it has a factor of 2. Thus the product of 2 even numbers is even because the product has a factor of 2. A similar argument holds for a set of multiples of 10. Now consider the odd whole numbers. An odd number is an even plus 1: O = E + 1. So the product of 2 odd numbers is: (E + 1) × (E + 1) = EE + (E + E) + 1. Now, EE is even and the sum of evens is even. So EE + E + E is even. Adding 1 to this sum results in an odd number. For parts *b* and *d* we see that 1 × 1 = 1 and 0 × 0 = 0.

38. **False:** consider the set of whole numbers: {0, 1} We see that $0 \times 0 = 0, 0 \times 1 = 0$ and $1 \times 1 = 1$. The set is finite and closed under multiplication.

39. a. $0 \div a = 0$ because this is equivalent to $0 = a \times 0$ which is true for all a not equal to 0 by the definition of the zero property.
    b. Suppose $a \div 0$ were defined and had a quotient c. We could rewrite it as a multiplication relation: $a \div 0 = c$ becomes $a = c \times 0 = 0$. But c could be any whole number and a quotient, by definition, is unique.
    c. Suppose $0 \div 0$ were defined and had a whole number quotient c. Then, by definition of division and quotient, c is a unique whole number. So $0 \div 0 = c$; or $0 = c \times 0$. But this would be true for all whole numbers. But the quotient of two whole numbers is unique. Thus $0 \div 0$ is undefined.

40. The responses to this question depend on previous activities and experiences of the students.

41. This question requires student interaction and activity.

42. Responses to the question will vary among students. They might include the following: The historical definition takes a multiplication relation which might be represented as (factor)(factor) = product and reforms it as the division relation: product ÷ factor = factor ÷ 1 = factor. This is a concept encountered repeatedly in the problems of this section.

43. To bring closure to the whole numbers under division we must expand the set of numbers to include the rational numbers. They are useful in situations in which a unit is to be distributed to a number of individuals. The whole numbers are a subset of the rational numbers. The quotient of any 2 whole numbers, if defined, exists within the set of rational numbers.

SECTION 2.4

1. a. **numeral**
   b. **number**
   c. **numerals**

2. a. Grouping the set by tens, one gets 1 group of 10 with 4 remaining elements. **14.**
   b. Grouping the set by 2's, 4's, and 8's, one gets 1 group of 8, 4, and 2, and none remaining: $\mathbf{1110_{two}}$.
   c. Grouping by fives, one gets two groups of 5 and 4 remaining: $\mathbf{24_{five}}$.

3. a. The number 586 would be represented with 5 'hundreds' squares, 8 'tens' sticks, and 6 'units' cubes.
   b. 4392 = **4(1000) + 3(100) + 9(10) + 2(1)**.
   c. 2,864,071 = $\mathbf{2(10^6) + 8(10^5) + 6(10^4) + 4(10^3) + 0(10^2) + 7(10^1) + 1(10^0)}$.

4. a. **3 groups of 100**.
   b. **3 groups of 25**.
   c. **3 groups of** $\mathbf{b^2}$.

5. a. 3(36) + 4(6) + 4(1) = **136**.
   b. 2(16) + 0(4) + 2(1) = **34**.
   c. 1(32) + 1(16) + 0(8) + 0(4) + 1(2) + 1(1) = **51**.

6. a. $256 = 1(6^3) + 1(6^2) + 0(6) + 4 = \mathbf{1104_{six}}$.
   b. $256 = 1(12^2) + 9(12) + 4 = \mathbf{194_{twelve}}$.
   c. $256 = 1(2^8) + 0(2^7) + 0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 0(2^2) + 0(2) + 0(1) = \mathbf{100000000_{two}}$.

7. a. $42_{seven} = 4(7) + 2 = 30_{ten} = 1(25) + 1(5) + 0 = \mathbf{110_{five}}$.
   b. $32_{four} = 3(4) + 2 = 14_{ten} = 2(6) + 2 = \mathbf{22_{six}}$.
   c. $1101100_{two} = 1(2^6) + 1(2^5) + 1(2^3) + 1(2^2) = \mathbf{108_{ten}} = 1(4^3) + 2(4^2) + 3(4^1) + 0(4^0) = \mathbf{1230_{four}} = 1(8^2) + 5(8^1) + 4(8^0) = \mathbf{154_{eight}}$.

8. The numbers, in order least to greatest are: **962, 980, 985, 1000, 2222, 2245**.

9. As described in the text: "...a tally system is based on establishing a one-to-one correspondence between a single mark and a single object so that the marks represent the number of objects. Later, grouping was used to simplify numeration systems."

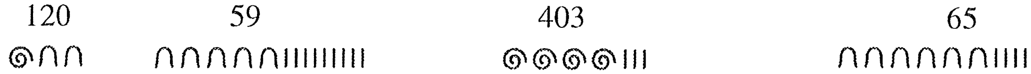
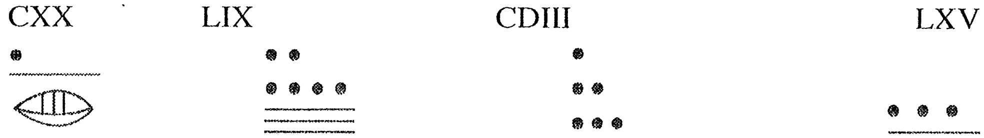
10. a. **1303**; b. **330**; c. **325**;

11. a. **1272**; b. **35**; c. **11,461**;

12. a. **1304**; b. **48**; c. **309**;

13. a. **126**; b. **92**; c. **510**.

14.

	120	59	403	65
a. Egyptian				
b. Babylonian				
c. Roman	CXX	LIX	CDIII	LXV
d. Mayan				

15. Various students will approach this question with different degrees of rigor.
    a. Egyptian numeration is based on a grouping system of powers of ten. One of each power of ten is represented by a different symbol and a number is represented with the fewest number of symbols. The Hindu-Arabic system is also based on powers of ten but is a place value rather than a grouping system.
    b. All students should note that both the Babylonian and Hindu-Arabic systems are place value systems although they use different bases. The Hindu-Arabic system has a different symbol for each numeral representing numbers from zero through ten. The Babylonian numeration system uses a modified tally and grouping system within the places.
    c. There are few similarities between the Roman and the Hindu-Arabic numeration systems. The Roman system is unique in that it utilizes both additive and subtractive relations. The Roman system also uses a grouping symbolism similar to the Egyptian.
    d. Hindu–Arabic has 10 symbols, but Mayan has only 3; Mayan uses tallying, but Hindu–Arabic doesn't.

16. See question 17 for a 10 by 10 hundreds chart. Various students will notice different patterns within the chart. Essentially all of the patterns are the result of the numeration system and the design of the chart. Completion of a row of the chart corresponds with the completion of counting through one power of the base.

17. BASE-TEN NUMERALS

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

a. To find 10 more than a number, go down one row in the same column.
b. To find 10 less than a number, go up one row in the same column.
c. To find one more than a number, move one space to the right or, if at the end of a row to the first space in the next row.
d. To find one less than a number, move one space to the left or if at the left end of the row, go to the far right space in the above row.
e. To find 11 more than a number, move one row down and one space right.
f. To find eleven less than a number, move one row up and one space left.
g. To find 9 more than a number, move one row down and then one space left.
h. To find 9 less than a number, move one row up and then one space right.

18. a. $4(5) + 4 = \mathbf{44_{five}}$
    b. $\mathbf{444_{five}}$
    c. This pattern occurs in the base-ten numeration system every time you have a number that is one less than a power of 10. It occurs with numbers such as 9; 99; 999; and 9,999.

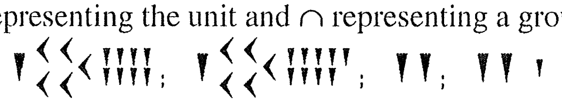
19. The numeral $rstu_b$ may be expanded to: $r(b^3) + s(b^2) + t(b^1) + u(b^0)$.

20. a. Because I travel I-55, a speed of 58 miles per hour is reasonable
    b. This semester I did spend about $85 on texts. Because this is about the same as last year, the difference is $0.
    c. Two (2) days is a reasonable time to begin to worry about a due assignment.
    d. My roommate, somewhat of a glutton, eats about 2 times as much as I.
    e. A blowout in basketball might be 85 to 52.
    f. Because checks are in packets of 25, a reasonable number to have remaining is 8.
    g. A candy sale fundraiser at a high school may be reasonably expected to raise $2500.
    h. If 25% of 20,000 students need a stamp to write home for money each week, the campus post office may expect to sell at least 5000 stamps each week.

21. a. The count of each set of symbols was multiplied by the value of the symbol. These products were then summed to obtain the base-ten numeral.
    b. The numerals in each place were summed as in part a and then these values were multiplied by the appropriate power of 60. Finally, these values were summed to obtain the base-ten value.
    c. The Roman numerals were grouped into sets of the same symbol and pairs of different symbols. If the different symbols decreased left to right their values were added. If the different symbols increased left to right the smaller was subtracted from the larger. Values of groups of the same symbol were multiplied by the number of symbols in the set. Finally, all these values were summed to obtain the base-ten representation.
    d. Determine the place of each symbol, multiply to determine the value of each symbol, and then add all the values.

22. a. 

    The Mayan system uses place value based on groups of 20 and a symbol for zero.

    b. 98, 99, 100, 101. The base-ten system uses a sequence of 10 symbols within a place.
    c. ∩∩∩∩∩∩|||||||||, ∩∩∩∩∩∩∩, ∩∩∩∩∩∩∩|. This is a grouping by tens system with representing the unit and ∩ representing a group of ten.
    d. 

    We have used a place value system, base 60, without a zero symbol.
    e. XCVIII, XCIX C, CI. This is the Roman system of part a.
    f. 69, 70, 71, 72. This is the base-ten system of part b.

23. It could be a square of 1000×1000 cm, or it could be a cube of $100 \times 100 \times 100$ cm.

24. BASE-TEN NUMERALS

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

BASE-FIVE NUMERALS

1	2	3	4	10	11	12	13	14	20
21	22	23	24	30	31	32	33	34	40
41	42	43	44	100	101	102	103	104	110
111	112	113	114	120	121	122	123	124	130
131	132	133	134	140	141	142	143	144	200
201	202	203	204	210	211	212	213	214	220
221	222	223	224	230	231	232	233	234	240
241	242	243	244	300	301	302	303	304	310
311	312	313	314	320	321	322	323	324	330
331	332	333	334	340	341	342	343	344	400

Among the generalizations students might form are: each column has a constant right hand digit, the place values advance more rapidly with base-five numerals than with base-ten numerals.

25. BASE-FIVE CHART

1	2	3		10
11	12	13	14	20
21	22	23	24	30
31	32	33	34	40
41	42	43	44	100

BASE-TEN CHART

SEE
THE BASE-TEN CHART
IN PROBLEM 13

The patterns identified in the two charts should be essentially the same.

26. The patterns predicted for the 8 by 8 base-eight chart should be essentially the same as those identified in the 5 by 5 base-five and the 10 by 10 base-ten charts in problem 21.

BASE-EIGHT CHART

1	2	3	4	5	6	7	10
11	12	13	14	15	16	17	20
21	22	23	24	25	26	27	30
31	32	33	34	35	36	37	40
41	42	43	44	45	46	47	50
51	52	53	54	55	56	57	60
61	62	63	64	65	66	67	70
71	72	73	74	75	76	77	100

27. The responses to this question depend upon the imaginations of the students.

28. The responses to this question depend on the insights into the structure of numeration systems that students have developed to this point in the course.

29. Assume that there are 10 hamburgers to a foot. Then 400 billion hamburgers would form a stack about 40 billion feet high. Assuming 5000 feet per a mile, the 40 billion (40 000 000 000) feet high stack is about 8 million miles high. This distance is about 32 times the earth-moon distance or about one twelfth the earth-sun distance. So the correct answer is **e**.

30. This debate on the merits of base-five versus base-ten depends on the insights that the students have gained into the structure of numeration systems.

31. Responses to this question depend on the prior mathematical and educational experiences of students and upon their willingness to do research.

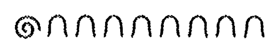
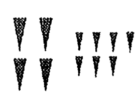
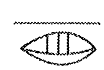
32. Responses to this question also depend upon the insights of students. One response should include the difficulty of using a place value system lacking a zero symbol.

33. The base-ten number represented by the numeral 421 could be represented as $\mathbf{645_{eight}}$; that is 6 groups of 64 plus 4 groups of 8 plus 5 units. If the base is to be 16, then additional symbols are required. A base 16 system similar to base ten requires 16 separate symbols. These commonly are, in counting order, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. Thus 421 = $\mathbf{1A5_{sixteen}}$. This numeral is I group of 256 plus A = 10_{ten} groups of 16 plus 5 units.

CHAPTER 2 REVIEW EXERCISES

1. a. The following sets have '7' in common: {#,#,#,#,#,#,#}. {<,<,<,<,<,<,<}, {*,*,*,*,*,*,*}, {$,$,$,$,$,$,$}, {1, 2, 3, 4, 5, 6, 7}.
   b. The sets { }, { }, { } show the meaning of 0.
   c. Let N = {$,$,$,$,$,$,$,$,$} and E = {@,@,@,@,@,@,@,@,@,@,@,@,@,@,@}. Since N is equivalent to a proper subset of E, n(N) < n(E) or 9 < 15, 15 > 9.
   d. Since { } is equivalent to a proper subset of any non-empty set, 0 < all other whole numbers.

2.

Hindu–Arabic	100	66	247	180
Egyptian				
Babylonian				
Roman	C	LXVI	CCXLVII	CLXXX
Mayan				

3. Responses will vary among students. Responses might include:
   a. The Egyptian numeration system is a tally system through 9 of a symbol with a different symbol for every power of 10. Early numeration had no 0 symbol.
   b. The Babylonian system is a place value system based on powers of 60. It incorporates a tally system of 2 symbols tallying units to 9 and tens to fifty.
   c. The Roman system uses different symbols for units, tens, fifties, hundreds, five hundreds, and thousands. It employs both additive and subtractive characteristics to keep the number of repetitions of a symbol to 3 or less.

d. The Mayan System is based on groups of 20, some tallying, place value, and 3 symbols, including a zero.
e. The Hindu-Arabic system is a place value system based on powers of 10. It uses different symbols for the numbers 0 through 9.

4. a. $184 = \mathbf{1214_{five}}$.
   b. $184 = \mathbf{10111000_{two}}$.

5. a, b. Student activities. For example, to represent 2045 use 2 thousands cubes, no flats, 4 sticks and 5 units.
   c. $2045 = \mathbf{2(1000) + 0(100) + 4(10) + 5(1)}$.
   d. $2045 = \mathbf{2(10^3) + 0(10^2) + 4(10^1) + 5(10^0)}$.

6. a. {2, 3, 4, 5, 6}
   b. {0, 1, 3, 4, 5, 9, 10, 11, 12}
   c. { }
   d. {2, 3, 4, 6}
   e. {0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 12}
   f. {2, 6}

7. Responses will vary. Examples are:
   a. I have 85 cm of heavy-gauge wire and 62 cm of fine-gauge wire. How much wire do I have?
   b. I have 85 National League baseball cards and 62 American League baseball cards. How many baseball cards do I have?

8. The responses will vary among students. Responses might be similar to:
   a. I have 85 inches of ribbon, and I plan to use 62 inches of it to decorate the archway for a wedding. How much ribbon is left to use on the bride's bouquet?
   b. There are 85 cookies in the box, and 62 of them are frosted. How many cookies are unfrosted?
   c. I have 85 feet of road frontage on my property. If I mow 62 feet of it on Saturday, how much is left to mow on Sunday?
   d. There are 85 students in the class, and 62 of them are currently earning passing grades. How many are not earning passing grades?
   e. I have two butterfly bushes in my yard. The one with the fence around it is 85 inches tall. The one without the fence around it has been eaten by deer and is 62 inches tall. How much taller is the bush with the fencing around it?
   f. There are 85 students in my 8:00 class and 62 students in my 2:00 class. How many more papers do I have to grade for my 8:00 class than for my 2:00 class?

9. Responses will vary. Examples are:
   a. There are 12 rows in my garden and each row is 25 feet long. How many feet of soaker hose do I need to water the entire garden at once?
   b. Each of 12 students brings 25 candies to class. How many candies were brought to class?
   c. A rectangle is 12 by 25 inches. What is the area of this rectangle?
   d. Joe has 12 shirts and 25 ties. How many different combinations of shirt and tie may he select from?

10. a. A teacher has a test bank of 125 questions. He wishes to construct examinations of 25 questions How many examinations may be constructed?
    b. If each student is to receive a 25 inch piece of ribbon from the 125 inches remaining on a roll, how many students will get ribbon?
    c. 125 pieces of colored paper are to be given to 25 art students. How many sheets of paper does each student get?
    d. A 125 inch piece of licorice is to be equally distributed among 25 students in a class. How long a piece does each student get?

11. The set of flowers may be represented {a,b,c,d}. A student may use some or all of the flowers, a subset of the set of flowers, but no student may use no flowers in an arrangement. There are 15 subsets of a 4 element set, excluding the null set. So the 15 students can each make a different arrangement.

12. $1045_b = \mathbf{1 \times b^3 + 0 \times b^2 + 4 \times b^1 + 5 \times b^0}$.

13. $180 - 59 = 121$ because $180 = 59 + 121$.

14. $123 > 85$ because $123 = 85 + 38$.

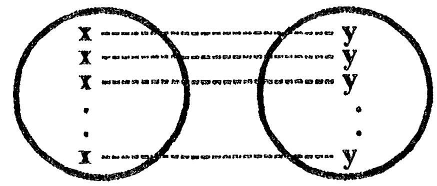
15. a. **S is closed** under addition because the sum of multiples of 10 is a multiple of 10.
    b. The **additive identity is 0** because any element of the set added to 0 results in the original element.
    c. The set is **closed under multiplication** because the set is multiples of 10 and if a multiple of 10 is multiplied by any whole number not 0, then the product contains a factor often and thus is a multiple of 10.
    d. The set **does not have** an element such that multiplication of any element by this elements results in the original element.
    e. The **associative, commutative, and distributive properties do apply** to S.

16. $120 \div 40 = 3$ because $3 \times 40 = 120$.

17. $125 \div 40 = 3 \text{ r } 5$ because $3 \times 40 + 5 = 125$.

18. Responses will vary among students. They might include the argument: Suppose that the situation modeled by the equation was: 18 children are to be taken on a trip in vehicles that can each carry 5 students. How many vehicles are required? In this situation 4 vehicles are required so the results of dividing 18 by 5 is reasonably 4. On the other hand, suppose 18 candies are to be equally shared among 5 students. Then each student receives 3 candies. This 18 divided by 5 is reasonably 3. Finally, 18 pieces of pizza are to be equally shared among 5 persons with the leftovers to be saved for breakfast. How many does each person eat and how much is left for breakfast results in a computation yielding 3 remainder 3.

19. The drawings will vary among students. They should represent the relation that sets are in a one-to-one correspondence if the elements of one set can be paired with elements of another set. If the pairing can produce pairs of identical elements the sets are called equal.



X **Y**

If each element of X can be paired be paired with an element of Y then the sets are in a 1 to 1 correspondence, represent the same whole number, and are equivalent. If, further, each element of X can be paired with an identical element in Y, the sets are equal.

20. Responses will vary. Students may suggest that the colors of the cords serve to distinguish place values and a single knot may represent 1 or 10 or 100 depending upon the color of the cord tied into the knot. The number of knots tied in the base cord might indicate the base used in that particular numeral.

21. Responses this question depend upon interactions within groups of students.

22. Responses to this questions depend upon the ingenuity of students.